

Risk Containment for Investors with Multivariate Utility Functions

Mark Kritzman, WCMB
mkritzman@wcmblc.com

Don Rich, Northeastern University and SSA
drich@cba.neu.edu, 617 234 9492

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RISK CONTAINMENT FOR INVESTORS WITH MULTIVARIATE UTILITY FUNCTIONS

Mark Kritzman

is a managing partner at Windham Capital Management—Boston in Cambridge, Massachusetts.

Don Rich

is professor of finance at the College of Business Administration at Northeastern University in Boston.

There are many financial situations in which investors care about joint occurrences, such as when: 1) a manager is evaluated against both an absolute target and a relative target; 2) an investor seeks protection from currency losses only when they coincide with unfavorable returns from the underlying portfolio; and 3) an investor wishes to structure an incentive fee that is conditioned on the simultaneous attainment of two objectives.

Conventional approaches to risk containment assume implicitly that investor utility depends on a single random variable, and that risk is defined as the variability of

this random variable. Investor behavior suggests, however, that investors care about multiple dimensions of risk.

This article develops a risk containment model in which investor utility is explicitly contingent on more than one random variable. The framework offers option-based hedging strategies that protect investors from the joint occurrence of negative outcomes. The model is also applied to incentive fees that are conditioned on more than one measure of performance. Finally, we combine these notions in order to engineer a hybrid collar that sacrifices concurrent favorable outcomes to finance protection from concurrent negative outcomes.

Traditional approaches to risk management seek to control the volatility of absolute returns (Markowitz [1952]), while other approaches focus on the variability of net returns (which is called tracking error); that is, a portfolio's returns relative to a benchmark's returns (Roll [1992]). There are many possible reasons, how-

ever, why a money manager may care concurrently about absolute and relative performance.¹

First, there may be economic considerations driven by consumption and competition. An endowment fund manager, for example, faces the dual objectives of funding the university's spending plan and maintaining the university's financial com-

petitiveness. In order to fund the university's spending plan, the manager must achieve a particular absolute return. Preserving the university's financial competitiveness, however, depends on the endowment fund's performance relative to the performance of institutions that compete for the same pool of faculty, students, and grants; hence relative returns are relevant as well.²

Second, the dual focus on absolute and relative performance may be motivated by agency issues. An investment advisor's compensation or job security may depend on relative performance, even if it has no direct economic consequence for the portfolio's constituents.³

Third, even in the absence of a formal evaluation criterion, natural competitiveness, self-esteem, or regret may lead investors to care about relative performance, even when economic consequences depend only on absolute performance (Clarke, Kruse, and Statman [1994]).

Finally, separate measures of performance may signal investment skill. Consider an investment advisor who invests in foreign markets. Although the financial performance of the fund depends on the combined performance of the local assets and the embedded currency exposure, a combined return of -5% representing a 15% local asset return and a 20% currency loss would probably be perceived more favorably than a -5% combined return representing a 0% local asset return and a -5% currency depreciation. In the latter case, there is no evidence of skill, while in the former the investment advisor has demonstrated skill in at least one decision.⁴ Rather than terminate the investment advisor, the client might simply retain the advisor and require that the currency exposure be hedged.

There are four possible outcomes for investors who care about two dimensions of risk. Exhibit 1 shows the four possibilities for investors who care about absolute and relative performance.

Obviously, it would be great for an investment advisor and its clients to achieve both a favorable absolute return and relative return (quadrant I). And although it would be disappointing to fall short of one of the targets, if the other target is met, this outcome may be viewed as at least tolerable (quadrant II or III).

The fourth possibility (quadrant IV), however,

EXHIBIT 1 MULTIPLE CONTINGENCY MATRIX

ABSOLUTE RETURNS	RELATIVE RETURNS	
	UNFAVORABLE	FAVORABLE
Favorable	III. Tolerable	I. Great
Unfavorable	IV. Disaster	II. Tolerable

is potentially intolerable. If an investment advisor fails to attain an absolute target and, at the same time, underperforms competing funds or a designated benchmark, it could be intolerable to the clients and potentially disastrous to the investment advisor. In short, we posit that managers are particularly averse to being *wrong and alone*.

Managers who display aversion to both absolute and relative risk can manage their exposure either through diversification or hedging. Chow [1995] has previously shown how to augment the Markowitz mean-variance paradigm to incorporate the two definitions of risk in the portfolio construction process. Chow argues that the bivariate diversification approach is supported by anecdotal evidence that investors commonly constrain the mean-variance solution so that it does not depart significantly from industry norms. For example, one might constrain the solution so that no more than 15% of total wealth is allocated to international equities and no more than 10% to real estate. We view this as evidence of investors' aversion to being *wrong and alone*.

Our purpose is to extend this multivariate description of utility to an option-based risk containment model.⁵ To do this, we first determine the equilibrium value of each quadrant of Exhibit 1. We develop a risk containment hedging model for investors with concave utility functions who are concerned with concurrent but distinct measures of risk.⁶ The formal model has a simple solution that can be used to guide portfolio protection strategies whether implemented using over-the-counter options or dynamic replication.⁷

We illustrate application of the general model to two specific protection strategies: 1) protection from the joint occurrence of a negative absolute return and a negative relative return, and 2) protection from currency losses that coincide with unfavor-

able local returns from the underlying foreign portfolio. The model reveals that protection from concurrent negative events is never more expensive and is almost always less expensive than protection from each negative event independently because of the diminished likelihood of concurrent negative events. The proposed risk containment strategies, therefore, should be particularly appealing to investors who can tolerate a negative event that occurs simultaneously with a positive event.

We also apply the multivariate model to incentive fees that are conditioned on more than one measure of performance. This duality is then shown to have important implications for the moral hazard risk that arises from univariate incentive fee arrangements.

Finally, traditional portfolio protection is commonly financed by an offsetting short option position. Financing the protection in this way is called a collar. We engineer a hybrid collar that finances protection from concurrent failures by relinquishing relative gains when the absolute return exceeds a particular threshold. We also show the sensitivity of this threshold to changes in the key variables.

I. VALUATION OF MULTIPLE-CONTINGENCY OPTIONS

We first present the general model, and then demonstrate how it is adapted to address the multiple contingencies represented by the quadrants in Exhibit 1.

The General Model

Exhibit 1 illustrates the general theme of the article, but the development of a formal model requires additional structure. Exhibit 2 presents an interpretation of the information in Exhibit 1. For expositional purposes, a zero absolute return demarcates favorable outcomes from unfavorable outcomes. A zero relative return is also used to distinguish favorable relative outcomes from unfavorable outcomes.⁸ The corresponding quadrant labels in Exhibit 1 are mapped in Exhibit 2.

The quadrant labels can also be defined more precisely. To do this, let the current time be defined as time zero and T be the end of the evaluation period. Let $S_B(t)$ and $S_p(t)$ denote, respectively, the value of the

EXHIBIT 2 GRAPHICAL REPRESENTATION OF THE MULTIPLE-CONTINGENCY MATRIX

	Absolute Returns	
III. Tolerable $S_p(T) \leq S_B(T)$ $S_p(T) \geq S_p$		I. Great $S_p(T) \geq S_B(T)$ $S_p(T) \geq S_p$
Option Payoff: $\text{Max}(S_B(T) - S_p(T), 0)$ if $S_p(T) \geq S_p$		Option Payoff: $\text{Max}(S_p(T) - S_B(T), 0)$ if $S_p(T) \geq S_p$
	0	Relative Returns
IV. Disaster $S_p(T) \leq S_B(T)$ $S_p(T) \leq S_p$		II. Tolerable $S_p(T) \geq S_B(T)$ $S_p(T) \leq S_p$
Option Payoff: $\text{Max}(S_B(T) - S_p(T), 0)$ if $S_p(T) \leq S_p$		Option Payoff: $\text{Max}(S_p(T) - S_B(T), 0)$ if $S_p(T) \leq S_p$

benchmark portfolio and the value of the managed portfolio at time t , $t \in [0, T]$, where $S_p(0) \equiv S_p$ and $S_B(0) \equiv S_B$. The initial portfolio values are assumed equal (i.e., $S_B = S_p$). Given this notation, it should be obvious, for example, that a favorable absolute holding-period return and favorable relative return (quadrant I) are characterized by $S_p(T)/S_p - 1 \geq S_B(T)/S_B - 1$ and $S_p(T)/S_p - 1 \geq 0$, which can be rewritten as $S_p(T) \geq S_B(T)$ and $S_p(T) \geq S_p$ as noted in Exhibit 2.

The next step to adding structure to the formulation is to determine the economic value of each quadrant. To do this, we assume (given the applications that follow) that investors desire the payoff from an option to exchange one asset for another if a particular state is realized. In other words, the economic value of quadrant IV is the current value of a European-style option that pays off: $\text{Max}(S_B(T) - S_p(T), 0)$ if $S_p(T) \leq S_p$. The option payoff for each of the quadrants is listed in Exhibit 2.

Exhibit 2 shows that determination of the economic value of each quadrant requires valuing four different options. To do this, we introduce general notation so that one model is applicable to all circumstances.

We want to value a default-free European exchange option, similar to Margrabe [1978] and Fischer [1978].⁹ Its payoff is contingent on satisfying two initial boundary conditions: one representing a restriction on relative returns — as in Margrabe and Fischer — and the other restricting absolute returns.

Let $W(S_j, S_i, S_k; \phi, H, T)$ represent the current value of a multiple-contingency option that pays off:

$$W(S_j(T), S_i(T), S_k(T); \phi, H, 0) \equiv \begin{cases} \text{Max}(S_j(T) - S_i(T), 0) & \text{if } \phi S_k(T) \geq \phi H \\ 0 & \text{if } \phi S_k(T) < \phi H \end{cases} \quad (1)$$

where $S_x(0) \equiv S_x$ for $x \in \{j, i, k\}$; ϕ is a binary variable that can be specified as 1 or -1 as desired; H is a constant denoting the hurdle level; the current time is defined to be time zero; and T is the expiration date of the option. It is, of course, possible that some of the delivery assets S_j , S_i , and S_k are identical assets. The option in Equation (1) can be viewed as a multiple-contingency call option on S_j struck at S_i or a put option on S_i struck at S_j .

We can simplify the distribution-free analysis of multiple-contingency options by identifying their relation to other options. As special cases, the multiple-contingency option in (1) can be seen to be isomorphic to 1) a Margrabe-Fischer exchange option when $\phi = 1$ and $H = 0$, and 2) an ordinary Black-Scholes call option when $\phi = 1$, $H = 0$, and $S_j(t)$ is a constant exercise price for all $t \in [0, T]$.

Consider engineering an asset with a current value of $E_j(t)$ that pays off $S_j(T)$ if $\phi S_k(T) \geq \phi H$ at $T \geq t$. It should be clear that the multiple-contingency option described in (1) can be viewed as an ordinary exchange option that pays off $\text{Max}(E_j(T) - E_i(T), 0)$. This implies that Margrabe's generalization to put-call parity also applies.

A multiple-contingency call option written on S_j and struck at S_i combined with an otherwise identical short put multiple-contingency option is equivalent

in value to the difference in the artificial assets:

$$W(S_j, S_i, S_k; \phi, H, T) - W(S_i, S_j, S_k; \phi, H, T) = E_j - E_i \quad (2)$$

Our next objective is to determine the exact partial equilibrium value for $W(S_j, S_i, S_k; \phi, H, T)$, which requires some distributional assumptions. It is assumed that $S_j(t)$, $S_i(t)$, and $S_k(t)$ denote the time t price of each traded asset whose return dynamics are described by the Itô process:

$$dS_x(t)/S_x(t) = \mu_x[t, S_x(t)]dt + \sigma_x dZ_x(t) \quad x \in \{j, i, k\} \quad (3)$$

where $\mu_x[t, S_x(t)]$ is the instantaneous expected return from holding asset x , and σ_x is the time-homogeneous local standard deviation rate of the logarithmic return for asset x . Moreover, $\{Z_x(t), x \in \{j, i, k\}\}$ is a standard Brownian motion with the trivariate distribution of $Z_j(t)$, $Z_i(t)$, and $Z_k(t)$ being normal with correlations ρ_{ij} , ρ_{ik} , and ρ_{jk} .¹⁰

In economic terms, the stochastic differential (3) says that price changes are assumed to follow a stationary random walk in continuous time. Under typical Black-Scholes assumptions, the equilibrium value of the option in (1) can be written as:

$$W(S_j, S_i, S_k; \phi, H, T) = S_j d_j^{-T} N_2(y_1, \phi y_2; \phi \bar{\rho}) - S_i d_i^{-T} N_2(y_1 - \sigma \sqrt{T}, \phi y_3; \phi \bar{\rho}) \quad (4)$$

where

$$y_1 \equiv \frac{\ln(S_j/S_i) + [\ln(d_i/d_j) + 1/2 \sigma^2]T}{\sigma \sqrt{T}}$$

$$y_2 \equiv \frac{\ln(S_k/H) + [\ln(r/d_k) - 1/2\sigma_k^2 + \sigma_k\sigma_j\rho_{j,k}]T}{\sigma_k\sqrt{T}}$$

$$y_3 \equiv \frac{\ln(S_k/H) + [\ln(r/d_k) - 1/2\sigma_k^2 + \sigma_k\sigma_i\rho_{i,k}]T}{\sigma_k\sqrt{T}}$$

and where $(r - 1)$ is the discrete riskless rate of interest on an investment maturing T years in the future, $(d_x - 1)$ is the payout rate of asset x ($x = j, i, k$), and $N_2(c_1, c_2; \rho)$ is the bivariate standard cumulative normal distribution function with upper limits of integration c_1 and c_2 and a correlation coefficient of ρ . Further, σ^2 , which is commonly referred to as the tracking error squared, is defined as the variance of the relative returns

$$\begin{aligned}\sigma^2 &\equiv \text{Var} \ln[S_j(T)/S_i(T)]/T \\ &= \sigma_i^2 + \sigma_j^2 - 2\sigma_i\sigma_j\rho_{i,j}\end{aligned}$$

and

$$\begin{aligned}\bar{\rho} &\equiv \text{Corr}[\ln S_k(T), \ln(S_j(T)/S_i(T))]T \\ &= (\sigma_j\rho_{j,k} - \sigma_i\rho_{i,k})/\sigma\end{aligned}$$

is the correlation of absolute and relative returns. A more detailed derivation of (4) is given in Appendix A.

Valuation of the Multiple-Contingency Quadrants

The general formula presented in (4) can easily be adapted to determine the value of the multiple contingencies represented by each of the four quadrants shown in Exhibit 1. Quadrant I represents the case in which the managed portfolio generates a positive absolute return and simultaneously exceeds the performance of the benchmark. This result is shown in the first row of Exhibit 3.

Quadrant II represents the case in which the managed portfolio generates a negative absolute

return but nonetheless exceeds the performance of the benchmark. Quadrant III represents the converse; that is, the managed portfolio produces a positive absolute return, but fails to exceed the benchmark's return. Finally, quadrant IV represents the scenario that we characterize as disastrous to the investment advisor: one in which both the absolute and relative performance are negative. These results appear in the second, third, and fourth rows of Exhibit 3.

II. APPLICATIONS OF THE MULTIPLE-CONTINGENCY OPTION

We adapt the general framework described in Exhibit 2 to address three specific portfolio management problems. First, we show how to protect a managed portfolio from underperforming a benchmark portfolio when the managed portfolio generates negative absolute returns. Second, we show how to protect an international portfolio from currency losses when the underlying foreign portfolio simultaneously generates negative local returns. Finally, to demonstrate that applications are not limited to risk containment issues, we apply our framework to value investment advisor incentive fees that are tied to relative and absolute measures of performance.

Contingent Exchange Options

Suppose an institutional investor wishes to protect a portfolio from underperforming a benchmark portfolio. Margrabe [1978] demonstrates that this type of portfolio protection is equivalent to an option to exchange one risky asset (the benchmark portfolio) for another (the managed portfolio). To see this result, notice that at the end of the holding period the combined value of the managed portfolio and European exchange option is $S_p(T) + \text{Max}(S_B(T) - S_p(T), 0)$ where $S_p(t)$ ($t \in [0, T]$) is the value of the managed portfolio, T is the end of the measurement period, and $S_B(t)$ is the benchmark portfolio value. Clearly, the investor receives the performance from the better-performing portfolio, thereby attaining the desired hedge.

As a cost savings alternative to this protection, suppose the investor decides that underperformance of the managed portfolio is tolerable as long as the portfolio does not depreciate in value. The investor's

EXHIBIT 3
VALUATION OF MULTIPLE-CONTINGENCY QUADRANTS

SCENARIO*	EUROPEAN OPTION PAYOFF	OPTION PRICING FORMULA**
I	Max($S_p(T) - S_B(T)$, 0) if $S_p(T) \geq H$	$S_p d_p^{-T} N_2(z_1, z_2, \bar{\rho}) - S_B d_B^{-T} N_2(z_1 - \sigma\sqrt{T}, z_3, \bar{\rho})$
II	Max($S_p(T) - S_B(T)$, 0) if $S_p(T) \leq H$	$S_p d_p^{-T} N_2(z_1, -z_2, -\bar{\rho}) - S_B d_B^{-T} N_2(z_1 - \sigma\sqrt{T}, -z_3, -\bar{\rho})$
III	Max($S_B(T) - S_p(T)$, 0) if $S_p(T) \geq H$	$S_B d_B^{-T} N_2(-z_1 + \sigma\sqrt{T}, z_3, -\bar{\rho}) - S_p d_p^{-T} N_2(-z_1, z_2, -\bar{\rho})$
IV	Max($S_B(T) - S_p(T)$, 0) if $S_p(T) \leq H$	$S_B d_B^{-T} N_2(-z_1 + \sigma\sqrt{T}, -z_3, \bar{\rho}) - S_p d_p^{-T} N_2(-z_1, -z_2, \bar{\rho})$

*Scenario I: Actively managed portfolio generates a positive return and outperforms the benchmark portfolio. Scenario II: Both portfolios generate negative returns, but the benchmark return is more negative. Scenario III: Both portfolios generate positive returns, but the benchmark portfolio generates a higher return. Scenario IV: Active portfolio generates a negative return and is outperformed by the benchmark.

**Where $z_1 \equiv (\ln(S_p/S_B) + [\ln(d_B/d_p) + 1/2 \sigma^2]T) / \sigma\sqrt{T}$, $z_2 \equiv (\ln(S_p/H) + [\ln(r/d_p) + 1/2 \sigma_p^2]T) / \sigma_p\sqrt{T}$, $z_3 \equiv (\ln(S_p/H) + [\ln(r/d_p) - 1/2 \sigma_p^2 + \sigma_p\sigma_B\rho_{p,B}]T) / \sigma_p\sqrt{T}$, $\bar{\rho} \equiv (\sigma_p - \sigma_B\rho_{p,B})/\sigma$, and $\sigma^2 \equiv \sigma_p^2 + \sigma_B^2 - 2\sigma_p\sigma_B\rho_{p,B}$. $N_2(c_1, c_2; \rho)$ is the bivariate standard cumulative normal distribution function with upper limits of integration c_1 and c_2 and a correlation coefficient of ρ . These results can be derived by making adjustments to Equation (4). For Scenario I, let $S_j \equiv S_p$, $S_i \equiv S_B$, $\phi \equiv 1$, $S_k \equiv S_p$, $\sigma_j \equiv \sigma_p$, $\sigma_i \equiv \sigma_B$, $\rho_{ij} \equiv \rho_{p,B}$, $\rho_{ik} \equiv \rho_{p,B}$, and $\rho_{jk} \equiv 1$. The notation for Scenario II is identical to Scenario I with the one exception that $\phi \equiv -1$. For Scenario III, let $S_j \equiv S_B$, $S_i \equiv S_p$, $\phi \equiv 1$, $S_k \equiv S_p$, $\sigma_j \equiv \sigma_B$, $\sigma_i \equiv \sigma_p$, $\rho_{ij} \equiv \rho_{p,B}$, $\rho_{ik} \equiv 1$, and $\rho_{jk} \equiv \rho_{p,B}$. The terms for Scenario IV are identical to Scenario III with the exception that $\phi \equiv -1$.

concern is underperforming the benchmark *and* generating a negative absolute return. The investor desires disaster insurance.

If both portfolios have the same initial notional value, the investor desires disaster protection in the form of an option that pays off $\text{Max}(S_B(T) - S_p(T), 0)$ if $S_p(T) \leq H$, which can be restated in terms of holding-period returns as: $\text{Max}(R_B(T) - R_p(T), 0)$ if $R_p(T) \leq 0$ where $R_B(T) = S_B(T)/S_B - 1$, $R_p(T) = S_p(T)/S_p - 1$, and $H = S_p$. Thus, by combining this option with the managed portfolio (i.e., $S_p(T) + \text{Max}(S_B(T) - S_p(T), 0)$ if $S_p(T) \leq S_p$), the investor receives the return on the benchmark portfolio if the active portfolio underperforms the benchmark and the absolute return is negative; otherwise the investor receives the return on the managed portfolio.

The fair market value of this disaster protection exchange option can be determined from Exhibit 3 (Scenario IV) or from Equation (4) as:¹¹

$$S_B d_B^{-T} N_2(y_1, -y_2; \bar{\rho}) -$$

$$S_p d_p^{-T} N_2(y_1 - \sigma\sqrt{T}, -y_3; \bar{\rho})$$

where:

$$y_1 \equiv \frac{\ln(S_B/S_p) + [\ln(d_p/d_B) + 1/2 \sigma^2]T}{\sigma\sqrt{T}}$$

$$y_2 \equiv \frac{\ln(S_p/H) + [\ln(r/d_p) + 1/2 \sigma_p^2 - \sigma_p\sigma\bar{\rho}]T}{\sigma_p\sqrt{T}}$$

$$y_3 \equiv \frac{\ln(S_p/H) + [\ln(r/d_p) + 1/2 \sigma_p^2]T}{\sigma_p\sqrt{T}}$$

the squared tracking error is $\sigma^2 = \sigma_p^2 + \sigma_B^2 -$

$2\sigma_B\sigma_P\rho_{B,P}$, and the correlation of the absolute and relative returns is $\bar{\rho} = (\sigma_P - \sigma_B\rho_{PB})/\sigma$.

To illustrate this application, let us assume that the portfolio value, the benchmark value, and the hurdle value are 100 on the valuation date; the riskless rate of interest is 6.0%; the portfolio's standard deviation is 15%; its tracking error with the benchmark is 3.0%; the correlation of the portfolio's absolute returns and the relative returns is 10.0%; and the time to expiration is one year.¹²

From Panel A of Exhibit 4 we see that with these assumptions the option value equals \$0.44. This value compares to a premium for an exchange option with no hurdle (i.e., $H = \infty$) of \$1.20. When the hurdle price is specified as an arbitrarily large value, in which case the contingency is always satisfied, the contingent exchange option collapses to the familiar Margrabe exchange option.¹³

It is also interesting to note the option value's sensitivity to the correlation between the portfolio's absolute and relative returns. The option value decreases with decreases in this correlation, because a low correlation reduces the likelihood that the portfolio will underperform its benchmark at the same time it experiences negative absolute returns. Holding constant the original inputs, Exhibit 4 shows that the option value falls from \$0.44 to \$0.33 as the correlation is reduced from 10% to -10%.

Finally (not shown in Exhibit 4), the effect of time to maturity on price is ambiguous. As time to maturity increases, the chance that the option will expire in the money rises, but time to maturity also increases the chance that the absolute hurdle will not be satisfied.

This contingent exchange option allows the investor to avoid the lower left quadrant of (quadrant IV) of Exhibit 2 by guaranteeing the benchmark performance when the portfolio experiences negative absolute returns. Its value is thus quantitatively dependent on the portfolio's relative return and qualitatively dependent on its absolute return.¹⁴

The option can also be modified to ensure that an investor avoids the lower left quadrant by guaranteeing an absolute return when the portfolio experiences negative relative returns. The value of this option would depend on the extent of the negative

return but not the extent of underperformance relative to the benchmark.

Contingent Currency Options

Investors who allocate a fraction of their portfolio to foreign assets simultaneously incur market risk and currency risk because the return from the foreign assets must be converted into the domestic currency. It is possible to develop risk containment strategies that simultaneously address market and currency risk of foreign investments.

Suppose an investor believes that a particular foreign equity market is undervalued but is uncertain about the direction of the country's currency. One way to act on this conviction would be to invest in the country's equity market and at the same time to purchase a European put option on the country's currency. This approach is attractive because it can be implemented with a simple buy-and-hold strategy. It will not necessarily eliminate currency risk completely, however, because if the portfolio goes up in foreign currency terms, its value may increase beyond the notional amount of the currency protection.

At the other extreme, an investor could eliminate currency risk completely by constantly adjusting the number of put options to coincide with the optimal hedge ratio. A third approach for gaining exposure to a foreign equity market without accepting the concomitant currency risk is to purchase a so-called *quanto option*, which is a fixed exchange rate foreign equity option in which the notional amount of the currency coverage expands or contracts to cover changes in the foreign currency value of the underlying portfolio.

A fourth and slightly different strategy is for the investor to seek protection from currency losses only when the underlying portfolio simultaneously generates negative local returns. This strategy implies that the investor is willing to accept currency losses as long as the underlying portfolio performs favorably. We refer to this strategy as a *contingent currency option*.

To formalize this particular notion, let $X(t)$ denote the exchange rate at time $t \in [0, T]$ (i.e., the price in dollars of one unit of the foreign currency). Let us also suppose that the investor wants currency protection below a constant critical level; call it C . Define $S(t)$ as the value of the managed portfolio and H as its hurdle price demarcating positive and nega-

EXHIBIT 4

A COMPARISON OF RESTRICTED AND UNRESTRICTED OPTION VALUES

PANEL A

σ_p (PER ANNUM)	$\bar{\rho}$ (PER ANNUM)	σ (PER ANNUM)	CONTINGENT EXCHANGE OPTION VALUE ^a	MARGRABE-FISCHER EXCHANGE OPTION VALUE ^a
0.15	0.10	0.03	\$0.44	\$1.20
0.15	-0.10	0.03	\$0.33	\$1.20
0.15	0.10	0.02	\$0.29	\$0.80
0.15	0.10	0.04	\$0.59	\$1.60

PANEL B

σ_s (PER ANNUM)	$\rho_{S,X}$ (PER ANNUM)	$\sigma = \sigma_X$ (PER ANNUM)	CONTINGENT CURRENCY OPTION VALUE ^b	GARMAN-KOHLHAGEN CURRENCY OPTION VALUE ^b
0.15	0.1	0.12	\$1.58	\$3.71
0.15	-0.1	0.12	\$1.22	\$3.71
0.15	0.1	0.10	\$1.27	\$2.96
0.15	0.1	0.14	\$1.90	\$4.46

PANEL C

σ_p (PER ANNUM)	$\bar{\rho}$ (PER ANNUM)	σ (PER ANNUM)	MULTICONDITIONAL INCENTIVE FEE ^c	UNRESTRICTED INCENTIVE FEE ^c
0.15	0.10	0.03	$\$0.76 \times 0.15 = \0.11	$\$1.20 \times 0.15 = \0.18
0.15	-0.10	0.03	$\$0.87 \times 0.15 = \0.13	$\$1.20 \times 0.15 = \0.18
0.15	0.10	0.02	$\$0.51 \times 0.15 = \0.08	$\$0.80 \times 0.15 = \0.12
0.15	0.10	0.04	$\$1.02 \times 0.15 = \0.15	$\$1.60 \times 0.15 = \0.24

^aThe contingent exchange option value is determined from section IIA, where $\sigma^2 = \sigma_p^2 + \sigma_B^2 - 2\sigma_p\sigma_B\rho_{p,B}$ and $\bar{\rho} = (\sigma_p - \sigma_B\rho_{p,B})/\sigma$, and the unrestricted option value is calculated from the Margrabe [1978] and Fischer [1978] formula to exchange the managed portfolio value for the benchmark. The parameter values for each option, unless otherwise specified, are $S_B = S_p = H = 100$, $r = 1.06$, and $d_B = d_p = T = 1$.

^bThe contingent currency option value is determined from section IIB, and the unrestricted option value is calculated from the Garman and Kohlhagen [1983] foreign currency option formula. The parameter values for each option, unless otherwise specified, are $X = S = C = H = 100$, $r_d = 1.06$, $r_f = 1.04$, $d_s = 1$, and $T = 1$.

^cThe multiconditional incentive fee value is determined from section IIC, where $\sigma^2 = \sigma_p^2 + \sigma_B^2 - 2\sigma_p\sigma_B\rho_{p,B}$ and $\bar{\rho} = (\sigma_p - \sigma_B\rho_{p,B})/\sigma$, and the unrestricted incentive fee value is calculated from the Margrabe [1978] formula to exchange the benchmark portfolio value for the managed portfolio. The parameter values for each option, unless otherwise specified, are $S_B = S_p = H = 100$, $r = 1.06$, $d_B = d_p = 1$, and $T = 1$.

tive returns. To hedge against a disaster scenario, the investor needs a European put option that pays off: $\text{Max}(C - X(T), 0)$ if $S(T) \leq H$, and zero otherwise.

The foreign currency version of Equation (4) can be written as:¹⁵

$$C r_d^{-T} N_2(y_1, -y_2; \rho_{S,X}) -$$

$$X r_f^{-T} N_2(y_1 - \sigma_X \sqrt{T}, -y_3; \rho_{S,X})$$

where

$$y_1 \equiv \frac{\ln(C/X) + [\ln(r_f/r_d) + 1/2\sigma_X^2]T}{\sigma_X\sqrt{T}}$$

$$y_2 \equiv \frac{\ln(S/H) + [\ln(r_f/d_S) - 1/2\sigma_S^2]T}{\sigma_S\sqrt{T}}$$

$$y_3 \equiv \frac{\ln(S/H) + [\ln(r_f/d_S) - 1/2\sigma_S^2 + \sigma_S\sigma_X\rho_{S,X}]T}{\sigma_S\sqrt{T}}$$

Here, r_f represents the foreign riskless rate, r_d denotes the domestic interest rate, and d_S is the dividend yield on portfolio S.

By setting the currency value, the currency strike price, the portfolio value, and the portfolio hurdle value all equal to 100 ($X = S = C = H = 100$), we protect the portfolio from currency losses whenever the underlying foreign assets generate negative returns in the local currency. The value at expiration of this contingent currency option does not depend on the magnitude of losses generated by the foreign assets in their local currency, but rather on the occurrence of the losses. Its value is determined by the extent of the currency losses along with the fact that the portfolio value denominated in the local currency is above or below the hurdle value.

Continuing with the example, we assume that the portfolio standard deviation is $\sigma_S = 15.0\%$, the currency standard deviation is $\sigma_X = 12.0\%$, the correlation between the portfolio returns and currency returns equals $\rho_{S,X} = -10.0\%$, the foreign and domestic interest rates equal 4.0% and 6.0% ($r_f = 1.04$, $r_d = 1.06$), and the time to expiration is one year. The value of the contingent currency option is \$1.22 as seen in Panel B of Exhibit 4. By contrast, the value of a currency option that pays off irrespective of the performance of the underlying foreign assets equals \$3.71, given the same inputs. Clearly, an investor who is concerned about currency losses only when the underlying assets perform poorly could achieve this protection much less expensively, at less than one-third the cost, than by purchase of an unconditional currency option.

The difference in value between the conditional and unconditional currency option values approaches zero as: 1) the hurdle on absolute returns goes to infinity, or 2) the managed portfolio and the exchange rate have the same cost of carry ($d_S = r_f$), their correlation goes to one, and the strike price equals the hurdle price ($C = H$). Another important difference between the conditional and unconditional currency option is that the conditional option value can decrease as a function of time to maturity. For example, the one year contingent currency option discussed above is worth \$1.22, while an otherwise identical five year contingent currency option is worth \$1.04.

Multiconditional Incentive Fees

Economic theory posits that managerial reward should correspond, at least in part, to the marginal product of management. In fact, many investment management fee arrangements are constructed with a base fee that is fixed and a contingent fee that is tied to the incremental return generated from a managed portfolio relative to a benchmark.¹⁶ If performance is valued appropriately, a manager should be indifferent between the “typical” flat management fee and the certainty equivalent of the combined value of the base fee and contingent fee. This conclusion, however, depends on the assumption that both parties are in agreement about the fair value of the incentive component.

Incentive fees that are based on the incremental return generated from a managed portfolio relative to a benchmark are effectively options to exchange one asset (the managed portfolio) for another (the benchmark portfolio). This characterization conceived by Margrabe [1978] as an application of his exchange option formula is explored in depth by Kritzman [1987]. While variations of the Margrabe model have been widely adopted by CTAs (commodity trading advisors), institutional investors have been considerably less enthusiastic about incentive fees.

CTAs are typically measured against an absolute return of 0% or the riskless rate of interest. In either case, they will not collect an incentive fee unless they generate profits for their customers. Managers of institutional funds, by contrast, are typically evaluated in terms of indexes, which may generate negative

returns. Thus, under an incentive fee arrangement, institutional fund managers could collect an incentive fee even if they produce losses for their clients. This fact may help to explain why most institutional investors lack enthusiasm for incentive fees.

The reluctance to embrace incentive fees on the part of some institutional investors might be overcome if incentive fees were tied to both relative and absolute measures of performance.¹⁷ We propose, for example, that the asymmetric portion of the incentive fee be set equal to the incremental return from the managed portfolio relative to the benchmark portfolio; that is, the contingent fee is α times $\text{Max}(S_p(T) - S_B(T), 0)$, where $\alpha(0 \leq \alpha \leq 1)$ is the manager's percentage of excess performance, $S_p(t)$ ($t \in [0, T]$) is the value of the managed portfolio, T is the end of the measurement period, and $S_B(t)$ is the value of the target portfolio.

The approach so far is equivalent to Margrabe's approach. Unlike Margrabe, however, we append the condition that absolute returns also be positive; specifically, that $S_p(T) \geq H$. The payoff from the incentive component part of the contract can then be stated formally as α times $\text{Max}(S_p(T) - S_B(T), 0)$ if $S_p(T) \geq H$ and zero otherwise.

The current value of this incentive fee is determined from Exhibit 1 (scenario I) or Equation (4) as:¹⁸

$$\alpha S_p d_p^{-T} N_2(y_1, y_2; \bar{\rho}) - \alpha S_B d_B^{-T} N_2(y_1 - \sigma \sqrt{T}, y_3; \bar{\rho})$$

where

$$y_1 \equiv \frac{\ln(S_p/S_B) + [\ln(d_B/d_p) + 1/2\sigma^2]T}{\sigma\sqrt{T}}$$

$$y_2 \equiv \frac{\ln(S_p/H) + [\ln(r/d_p) + 1/2\sigma_p^2]T}{\sigma_p\sqrt{T}}$$

$$y_3 \equiv \frac{\ln(S_p/H) + [\ln(r/d_p) + 1/2\sigma_p^2 + \sigma_p\sigma\bar{\rho}]T}{\sigma_p\sqrt{T}}$$

σ^2 depicts the variance in relative returns $\sigma^2 \equiv \sigma_p^2 + \sigma_B^2 - 2\sigma_B\sigma_p\rho_{PB}$, and $\bar{\rho}$ takes the form $\bar{\rho} \equiv (\sigma_p - \sigma_B\rho_{B,P})/\sigma$.

To illustrate how this formula is used to value an incentive fee contract, assume the same inputs that are used in the contingent exchange option example and that the manager's incentive participation is 15% of positive excess performance.¹⁹ These values, as shown in Panel C of Exhibit 4, imply that the incentive component of the fee is worth 11 basis points to the manager. Thus, if the manager would otherwise charge a flat fee of 100 basis points to manage the portfolio, the base component of the incentive fee should be set equal to 89 basis points. This difference compensates for the value of the option provided to the manager by the incentive arrangement.

If the base component were set higher than 89 basis points, the manager would receive an unfair advantage by forgoing less of the alternative flat fee than the incentive component is worth. Conversely, the manager would be disadvantaged if the base component were set below 89 basis points.

The value to the manager of the incentive component is less than it would be were the incentive component not conditioned on the joint occurrence of outperforming the benchmark *and* producing a positive absolute return. For example, if we remove the contingency on absolute returns but leave the other assumptions unchanged, the value to the manager of the incentive component rises to 18 basis points, implying that the fair value of the base component declines to 82 basis points. The obvious intuition behind these changes is that the manager is more likely to collect the incentive component when the absolute hurdle is removed.

The multiconditional approach to incentive fees helps to mitigate another limitation of the standard incentive fee arrangement. A moral hazard arises with standard incentive fees because the manager can increase the value of the incentive component by changing the managed portfolio's risk relative to the benchmark.²⁰ To discourage this type of behavior many incentive arrangements limit the maximum payoff the manager may receive.²¹

The imposition of a cap, however, introduces yet another problem. When a manager reaches the cap, it is in the manager's best interest to lock in the

incentive component by pursuing a passive strategy for the remainder of the evaluation period.

By conditioning the incentive component on the joint performance of absolute and relative returns, the relation between portfolio volatility and the value of the incentive component is ambiguous. While under the standard incentive fee arrangement, the manager can increase tracking error (and therefore the value of the fee) simply by moving the portfolio's volatility away from the benchmark's volatility, under the multiconditional incentive fee arrangement, a change in portfolio volatility affects the value of the incentive component in several ways.

First of all, portfolio volatility directly affects the value of the incentive component, because it influences the probability that the portfolio will exceed the hurdle value. The direction of this effect, however, can be positive or negative. Moreover, portfolio volatility affects the value of the incentive component indirectly through its impact on tracking error and the correlation of absolute and relative returns. These indirect effects sometimes offset the direct effect and sometimes amplify the direct effect.

The important point, though, is that the net effect of a change in portfolio volatility on the value of the incentive component is obscured by the duality of the multiconditional incentive fee. Thus, any attempt by a manager to game the incentive fee could easily backfire. This issue is addressed more thoroughly in Appendix B.

III. A HYBRID COLLAR TO FINANCE DISASTER PROTECTION

We have shown how to value options that protect investors from concurrent negative outcomes and how to structure an incentive fee whose payoff is contingent on the joint attainment of an absolute and relative target. Now we combine these two notions to demonstrate how protection from concurrent negative events can be financed by forgoing relative gains that occur when the absolute gain simultaneously exceeds a given threshold.

Many investors face an asymmetric reward system in that success is rewarded less than failure of an equal magnitude is reproved. These investors are therefore willing to sacrifice upside participation for

downside protection by investing in hedging strategies known as collars. A zero-cost collar combines the sale of a call option with the purchase of a put option. The premium received from the call option, which limits upside participation in the underlying asset, exactly finances the premium for the put option, which establishes the floor.

We wish to apply this notion to the disaster option; that is, we wish to finance protection from the joint occurrence of an absolute and relative failure by sacrificing relative gains that occur when the absolute performance exceeds a particular hurdle. A zero-cost collar is found by solving for an absolute hurdle value associated with a call option on joint favorable performance (quadrant I, Exhibit 1) so that its premium equals the premium for the disaster option (quadrant IV, Exhibit 1).²² Specifically, we wish to solve for H so that the current value of a European option that pays off $\text{Max}(S_p(T) - S_B(T), 0)$ if $S_p(T) \geq H$ equals the current value of an option that pays off $\text{Max}(S_B(T) - S_p(T), 0)$ if $S_p(T) \leq S_p$.

The appeal of such a strategy, of course, depends on the level of the hurdle above which relative gains are relinquished. Exhibit 5A shows that, for a reasonably wide range of assumptions, the cushion on favorable absolute performance that must be exceeded before the investor is required to relinquish relative gains is close to 15%, as long as the riskless return is 6% and the holding period is one year. This result is apparently surprising, according to the informal surveys we have conducted and remarkably robust to all the inputs except the riskless return and time to maturity.

Our analysis also reveals that the trade-off between the cushion and the deductible is not symmetric. Exhibit 5B shows the results for a hybrid collar in which the joint downside protection is guaranteed only when the absolute return falls below -5%. (In this case, the disaster option pays off $\text{Max}(S_B(T) - S_p(T), 0)$ if $S_p(T) \leq 0.95S_p$.) Clearly, the cushion on favorable absolute performance is higher because the collar finances less protection, but it is interesting to note that the increase in the cushion is nearly three times the magnitude of the increase in the deductible.

It is possible to engineer other collars that integrate the other quadrants of the multiple-contingency matrix in Exhibit 1. Moreover, it is possible to struc-

EXHIBIT 5A**HYBRID COLLAR WITH 0% ABSOLUTE TARGET**

σ_p	σ	$\bar{\rho}$	T	r	H	CUSHION = (H/100) - 1	OPTION VALUE
0.15	0.05	0.1	1.0	1.06	114.83	14.83%	\$0.734
0.10	0.05	0.1	1.0	1.06	113.43	13.43%	\$0.611
0.20	0.05	0.1	1.0	1.06	116.83	16.83%	\$0.789
0.15	0.02	0.1	1.0	1.06	114.88	14.88%	\$0.293
0.15	0.10	0.1	1.0	1.06	114.74	14.74%	\$1.472
0.15	0.05	0.0	1.0	1.06	114.92	14.92%	\$0.641
0.15	0.05	-0.1	1.0	1.06	115.00	15.00%	\$0.552
0.15	0.05	0.1	0.5	1.06	107.16	7.16%	\$0.593
0.15	0.05	0.1	5.0	1.06	199.66	99.66%	\$0.811
0.15	0.05	0.1	1.0	1.04	110.54	10.54%	\$0.832
0.15	0.05	0.1	1.0	1.08	119.21	19.21%	\$0.643

EXHIBIT 5B**HYBRID COLLAR WITH -5% ABSOLUTE TARGET**

σ_p	σ	$\bar{\rho}$	T	r	H	CUSHION = (H/95) - 1	OPTION VALUE
0.15	0.05	0.1	1.0	1.06	120.88	27.24%	\$0.495
0.10	0.05	0.1	1.0	1.06	119.40	25.68%	\$0.307
0.20	0.05	0.1	1.0	1.06	122.98	29.45%	\$0.600
0.15	0.02	0.1	1.0	1.06	120.93	27.29%	\$0.198
0.15	0.10	0.1	1.0	1.06	120.78	27.14%	\$0.994
0.15	0.05	0.0	1.0	1.06	120.96	27.33%	\$0.419
0.15	0.05	-0.1	1.0	1.06	121.06	27.43%	\$0.348
0.15	0.05	0.1	0.5	1.06	112.80	18.74%	\$0.347
0.15	0.05	0.1	5.0	1.06	210.16	121.22%	\$0.643
0.15	0.05	0.1	1.0	1.04	116.36	22.48%	\$0.579
0.15	0.05	0.1	1.0	1.08	125.48	32.08%	\$0.420

ture still other hybrid collars so that only relative gains above a specified level are relinquished to finance the protection. In this case, the investment advisor would still experience some relative outperformance while achieving protection from concurrent failures.

IV. SUMMARY

We have derived an option-based framework to protect investors from the simultaneous occurrence of negative events. This framework implies that investor utility depends on the joint behavior of more than one random variable. We also extend the framework to value incentive fees that are linked to the

attainment of two or more distinct but potentially correlated goals. This framework obscures the relation between the managed portfolio's volatility and the value of the incentive component, thereby diminishing the likelihood that the fee will be gamed. Finally, we combine these valuations to engineer a hybrid collar that provides protection from joint negative events in exchange for relinquishing relative gains that occur above an absolute threshold.

Our analysis demonstrates that the value of the contingent options and the value of the multiconditional incentive fees depend importantly on the correlation of the random variables. As the correlation falls, investors who can tolerate an unfavorable out-

come that is accompanied by a favorable outcome should find this type of protection more and more attractive. Moreover, investment clients who pay incentive fees based on the concurrent attainment of multiple targets should be increasingly willing to pay a higher base fee than they would otherwise pay under an unconditional incentive fee arrangement.

APPENDIX A CONFIRMATION OF EQUATION (4)

Observe that the function $W(S_j, S_i, S_k; \phi, H, T)$ is linear homogeneous of degree one with respect to S_j and S_i ; therefore, let $W(S_j, S_i, S_k; \phi, H, T) \equiv S_i w(S, S_k; 1, \phi, H, T)$ where $S(t) \equiv S_j(t)/S_i(t) \forall t \in [0, T]$ and $w(\cdot)$ is a European option that pays off $\text{Max}(S(T) - 1, 0)$ if $\phi S_k(T) \geq \phi H$ and zero otherwise.

If markets are assumed complete, a change of the probability measure from Girsanov's theorem allows us to rewrite the stochastic differential in (3) as:

$$dS_x(t)/S_x(t) = \ln(r/d_x)dt + \sigma_x d\tilde{Z}_x(t) \quad x \in \{j, i, k\}$$

where $(r - 1)$ is the discrete riskless rate of interest on an investment maturing T years in the future, and $(d_x - 1)$ is the payout rate of asset x ($x = j, i, k$).

Under the assumptions noted, if the capital markets are also frictionless, standard hedging arguments can be used to show that $W(S_j, S_i, S_k; \phi, H, T)$ is the solution to the second-order partial differential equation subject to the boundary conditions noted above in (1):

$$\begin{aligned} \ln(r)W(S_j, S_i, S_k; \phi, C, T) &= \ln(r/d_j)S_j W_{S_j} + \\ \ln(r/d_k)S_k W_{S_k} &+ \ln(r/d_i)S_i W_{S_i} - W_T + \\ 1/2S_j^2\sigma_j^2 W_{S_j S_j} &+ 1/2S_k^2\sigma_k^2 W_{S_k S_k} + \\ 1/2S_i^2\sigma_i^2 W_{S_i S_i} &+ S_j S_k \sigma_j \sigma_k \rho_{k,j} W_{S_j S_k} + \\ S_j S_i \sigma_j \sigma_i \rho_{i,j} W_{S_j S_i} &+ S_i S_k \sigma_k \sigma_i \rho_{i,k} W_{S_k S_i} \end{aligned}$$

where the subscripts denote partial derivatives.

Further, it is easy to show that the value of $w(\cdot)$ is the solution to the second-order differential equation

$$\begin{aligned} \ln(d_i)w &= \ln(d_i/d_j)S w_{S_j} + (\ln(r/d_k) + \sigma_i \sigma_k \rho_{i,k}) \times \\ &S_k w_{S_k} - w_T + 1/2S^2\sigma_s^2 w_{SS} + \\ &1/2S_k^2\sigma_k^2 w_{S_k S_k} + S S_k \sigma_s \sigma_k \bar{\rho} w_{SS_k} \end{aligned}$$

This result follows by substitution (i.e., $W_j = w_s$, $W_k = S_j w_k$, $W_i = w - S w_s$, $W_{jj} = w_{ss}/S_i$, $W_{kk} = w_{kk} S_i$, $W_{ii} = S^2 w_{ss}/S_i$, $W_T = S_i w_T$, $W_{jk} = w_{sk}$, $W_{ji} = -S w_{ss}/S_i$, and $W_{ik} = w_k - S w_{sk}$). If the risk-adjusted drift rate of asset k is set equal to $\ln(r/d_k) + \sigma_k \sigma_i \rho_{i,k}$, the discount rate (under the equivalent martingale measure) of the relative asset S is d_i whose proportional variance is $\sigma_s^2 = \sigma_i^2 + \sigma_j^2 - 2\sigma_i \sigma_j \rho_{i,j}$ and payout rate is d_i . Further, observe that the correlation coefficient of the relative asset and asset k is $\bar{\rho} = (\sigma_j \rho_{j,k} - \sigma_i \rho_{i,k})/\sigma_s$. Thus, the solution to this new partial differential equation is immediately determined from Rich and Leipus [1997, theorem 5] as

$$w(S, S_k; \phi, H, 1, T) = S d_j^{-T} N_2(y_1, \phi y_2; \phi \bar{\rho}) -$$

$$d_i^{-T} N_2(y_1 - \sigma_s \sqrt{T}, \phi y_3; \phi \bar{\rho})$$

where

$$y_1 \equiv \frac{\ln(S) + [\ln(d_i/d_j) + 1/2\sigma_s^2]T}{\sigma_s \sqrt{T}}$$

$$y_2 \equiv \frac{\ln(S_k/H) + [\ln(r/d_k) - 1/2\sigma_k^2 + \sigma_k \sigma_i \rho_{i,k} + \sigma_k \sigma_s \bar{\rho}]T}{\sigma_k \sqrt{T}}$$

$$y_3 \equiv \frac{\ln(S_k/H) + [\ln(r/d_k) - 1/2\sigma_k^2 + \sigma_k \sigma_i \rho_{i,k}]T}{\sigma_k \sqrt{T}}$$

Equation (4) is then confirmed by multiplying through by S_i and substituting in $\sigma_k \sigma_s \bar{\rho} = \sigma_k \sigma_j \rho_{j,k} - \sigma_k \sigma_i \rho_{i,k}$.

The next step is to analyze the hedging properties of the option by determining the comparative statics. The comparative statics for the option pricing function can, of course, also be substituted into the partial differential equation above to confirm that it is the unique equilibrium solution.

To conserve space, let

$$y_1^* \equiv y_1 - \sigma_s \sqrt{T}, \quad \rho^* \equiv \frac{\bar{\rho}}{\sqrt{1 - \bar{\rho}^2}},$$

$$a \equiv \frac{y_1 - \bar{\rho}y_2}{\sqrt{1 - \bar{\rho}^2}}, \quad b \equiv \frac{\phi y_2 - \phi \bar{\rho} y_1}{\sqrt{1 - \bar{\rho}^2}},$$

$$c \equiv \frac{y_1^* - \bar{\rho}y_3}{\sqrt{1 - \bar{\rho}^2}}, \quad d \equiv \frac{\phi y_3 - \phi \bar{\rho} y_1^*}{\sqrt{1 - \bar{\rho}^2}}$$

Further, let $N(x)$ denote the standard cumulative normal distribution function and $n(x)$ denote the standard normal density function. Note that the partial derivatives below are defined only when $S_j(T) > S_i(T)$ and $\phi S_k(T) > \phi H$. The partial derivatives confirm that the partial differential equation is satisfied by noting: .

$$n(x)n\left(\frac{y - \rho x}{\sqrt{1 - \rho^2}}\right) = n(y)n\left(\frac{x - \rho y}{\sqrt{1 - \rho^2}}\right)$$

$$\frac{\partial(4)}{\partial S_j} = d_j^{-T} N_2(y_1, \phi y_2; \phi \bar{\rho}) + d_j^{-T} N(b)n(y_1)/[\sigma_s \sqrt{T}] -$$

$$S_i d_i^{-T} N(d)n(y_1^*)/[S_j \sigma_s \sqrt{T}]$$

$$\frac{\partial(4)}{\partial S_i} = -S_j d_j^{-T} N(b)n(y_1)/[S_i \sigma_s \sqrt{T}] -$$

$$d_i^{-T} N_2(y_1^*, \phi y_3; \phi \bar{\rho}) + d_i^{-T} N(d)n(y_1^*)/[\sigma_s \sqrt{T}]$$

$$\frac{\partial(4)}{\partial S_k} = \phi S_j d_j^{-T} N(a)n(y_2)/[S_k \sigma_k \sqrt{T}] -$$

$$\phi S_i d_i^{-T} N(c)n(y_3)/[S_k \sigma_k \sqrt{T}]$$

$$\frac{\partial^2(4)}{\partial S_j^2} = d_j^{-T} N(b)n(y_1)/[S_j \sigma_s \sqrt{T}] - \phi d_j^{-T} n(b)n(y_1)\rho^*/[S_j \sigma_s^2 T] -$$

$$d_j^{-T} N(b)n(y_1)y_1/[S_j \sigma_s^2 T] + S_i d_i^{-T} N(d)n(y_1^*)/[S_j^2 \sigma_s \sqrt{T}] +$$

$$\phi S_i d_i^{-T} n(d)n(y_1^*)\rho^*/[S_j^2 \sigma_s^2 T] + S_i d_i^{-T} N(d)n(y_1^*)y_1^*/[S_j \sigma_s^2 T]$$

$$\frac{\partial^2(4)}{\partial S_i^2} = S_j d_j^{-T} N(b)n(y_1)/[S_i^2 \sigma_s \sqrt{T}] - \phi S_j d_j^{-T} n(b)n(y_1)\rho^*/[S_i^2 \sigma_s^2 T] -$$

$$S_j d_j^{-T} N(b)n(y_1)y_1/[S_i^2 \sigma_s^2 T] + d_i^{-T} N(d)n(y_1^*)/[S_i \sigma_s \sqrt{T}] +$$

$$\phi d_i^{-T} n(d)n(y_1^*)\rho^*/[S_i \sigma_s^2 T] + d_i^{-T} N(d)n(y_1^*)y_1^*/[S_i \sigma_s^2 T]$$

$$\frac{\partial^2(4)}{\partial S_k^2} = -\phi S_j d_j^{-T} N(a)n(y_2)/[S_k^2 \sigma_k \sqrt{T}] - \phi S_j d_j^{-T} n(a)n(y_2)\rho^*/[S_k^2 \sigma_k^2 T] -$$

$$\phi S_j d_j^{-T} N(a)n(y_2)y_2/[S_k^2 \sigma_k^2 T] + \phi S_i d_i^{-T} N(c)n(y_3)/[S_k^2 \sigma_k \sqrt{T}] +$$

$$\phi S_i d_i^{-T} n(c)n(y_3)\rho^*/[S_k^2 \sigma_k^2 T] + \phi S_i d_i^{-T} N(c)n(y_3)y_3/[S_k^2 \sigma_k^2 T]$$

$$\frac{\partial^2(4)}{\partial S_j \partial S_i} = -d_j^{-T} N(b)n(y_1)/[S_i \sigma_s \sqrt{T}] + \phi d_j^{-T} n(b)n(y_1)\rho^*/[S_i \sigma_s^2 T] +$$

$$d_j^{-T} N(b)n(y_1)y_1/[S_i \sigma_s^2 T] - d_i^{-T} N(d)n(y_1^*)/[S_j \sigma_s \sqrt{T}] -$$

$$\phi d_i^{-T} n(d)n(y_1^*)\rho^*/[S_j \sigma_s^2 T] - d_i^{-T} N(d)n(y_1^*)y_1^*/[S_j \sigma_s^2 T]$$

$$\frac{\partial^2(4)}{\partial S_k \partial S_i} = [S_i d_i^{-T} n(c)n(y_3) - S_j d_j^{-T} n(b)n(y_1)]/$$

$$(\phi S_k S_i \sigma_s \sigma_k T \sqrt{1 - \bar{\rho}^2}) - \phi d_i^{-T} N(c)n(y_3)/[S_k \sigma_k \sqrt{T}]$$

$$\frac{\partial^2(4)}{\partial S_k \partial S_j} = \phi d_j^{-T} N(a)n(y_2)/[S_k \sigma_k \sqrt{T}] + (\phi d_j^{-T} n(a)n(y_2)/$$

$$[S_k - \phi S_i d_i^{-T} n(d)n(y_1^*)/S_j \sigma_k]/(\sigma_k \sigma_s T \sqrt{1 - \bar{\rho}^2})$$

$$\frac{\partial(4)}{\partial T} = \phi S_j d_j^{-T} N(a)n(y_2)/\sqrt{T} \times$$

$$[-1/2 y_2/\sqrt{T} + \ln(r/d_k)/\sigma_k - 1/2 \sigma_k + \sigma_j \rho_{k,j}] +$$

$$\ln(d_i) S_i d_i^{-T} N_2(y_1^*, \phi y_3; \phi \bar{\rho}) + S_j d_j^{-T} N(b)n(y_1) \times$$

$$[-1/2 y_1/T + \ln(d_i/d_j)/[\sigma_s \sqrt{T}] + 1/2 \sigma_s/\sqrt{T}] -$$

$$\ln(d_j)S_j d_j^{-T} N_2(y_1, \phi y_2; \bar{\rho}) - S_i d_i^{-T} (\phi N(c)n(y_3) \times$$

$$[-1/2 y_3 / \sqrt{T} + \ln(r/d_k) / \sigma_k - 1/2 \sigma_k + \sigma_i \rho_{i,k}] +$$

$$N(d)n(y_1^*) [-1/2 y_1 / \sqrt{T} + \ln(d_i/d_j) / \sigma_s] / \sqrt{T}$$

APPENDIX B

ANALYSIS OF MORAL HAZARD RISK WITH MULTICONDITIONAL INCENTIVE FEES

The value of the multiconditional incentive fee changes in a complex manner with respect to changes in the risk of the managed portfolio. A change in σ_p affects the incentive fee value directly and, in two ways, indirectly. The incentive fee is 1) a function of σ_p (direct effect), 2) a function of the tracking error, σ , which is a function of σ_p (indirect effect), and 3) a function of the correlation of absolute and relative returns, $\bar{\rho}$, which is a function of σ and σ_p (indirect effect). Therefore, a rate of change analysis requires the examination of the total derivative. To do this, we write

$$\frac{dW(.)}{d\sigma_p} = \frac{\partial W(.)}{\partial \sigma_p} + \frac{\partial W(.)}{\partial \sigma} \frac{d\sigma}{d\sigma_p} + \frac{\partial W(.)}{\partial \bar{\rho}} \frac{d\bar{\rho}}{d\sigma_p}$$

$$\text{Total} = \text{Direct} + \text{Indirect Effect} + \text{Indirect}$$

$$\text{Effect} \quad \text{Effect} \quad \text{from } \sigma \quad \text{Effect from } \bar{\rho}$$

Because changes in $\bar{\rho}$ with respect to changes in σ_p come from σ_p and σ , we must state the last term on the right-hand side as another total derivative:

$$\frac{d\bar{\rho}}{d\sigma_p} = \frac{\partial \bar{\rho}}{\partial \sigma_p} + \frac{\partial \bar{\rho}}{\partial \sigma} \frac{d\sigma}{d\sigma_p}$$

This result is interpreted as follows. Changes in σ_p change the value of the incentive fee through its indirect effect on $\bar{\rho}$. The indirect effect of a change in the incentive fee value from a change in $\bar{\rho}$ comes directly from the fact that $\bar{\rho}$ is a function of σ_p (the indirect-direct effect) and indirectly from the fact that $\bar{\rho}$ is a function of σ that is a function of σ_p (the indirect-indirect effect).

Incorporating this total derivative into the previous equation renders:

$$\frac{dW(.)}{d\sigma_p} = \frac{\partial W(.)}{\partial \sigma_p} + \frac{\partial W(.)}{\partial \sigma} \frac{d\sigma}{d\sigma_p} + \frac{\partial W(.)}{\partial \bar{\rho}} \frac{d\bar{\rho}}{d\sigma_p} + \frac{\partial W(.)}{\partial \bar{\rho}} \frac{\partial \bar{\rho}}{\partial \sigma} \frac{d\sigma}{d\sigma_p}$$

$$\text{Total} = \text{Direct} + \text{Indirect Effect} + \text{Indirect} + \text{Indirect Effect}$$

$$\text{Effect} \quad \text{Effect} \quad \text{from } \sigma \quad \text{Effect from} \quad \text{from } \bar{\rho}$$

$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{from } \sigma_p \quad \text{from } \sigma$$

Because

$$\frac{d\sigma}{d\sigma_p} = \bar{\rho}, \quad \frac{\partial \bar{\rho}}{\partial \sigma_p} = \frac{1}{\sigma}, \quad \text{and} \quad \frac{\partial \bar{\rho}}{\partial \sigma} = -\frac{\bar{\rho}}{\sigma}$$

we can rewrite the total derivative as:

$$\frac{dW(.)}{d\sigma_p} = \frac{\partial W(.)}{\partial \sigma_p} + \frac{\partial W(.)}{\partial \sigma} \bar{\rho} + \frac{\partial W(.)}{\partial \bar{\rho}} \frac{(1 - \bar{\rho}^2)}{\sigma}$$

$$(?) = (\pm) + (+) (\pm) + (\pm) \quad (+)$$

We conclude from this result that the total change in the value of the multiconditional incentive fee that arises from a change in the risk of the managed portfolio is virtually impossible to anticipate. Depending on the other parameters, the change σ_p that is directly transmitted to the incentive fee can increase or decrease its value. Moreover, the indirect effect of a change in σ_p that changes σ that alters the value of the incentive fee is positive (negative) if $\bar{\rho}$ is positive (negative). Finally, the indirect effect of a change in σ_p on $\bar{\rho}$, may also increase or decrease the incentive fee value depending on the other inputs. This recondite relation between volatility and the value of the incentive fee component should significantly diminish moral hazard risk.

ENDNOTES

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¹If an investor focuses on *either* the volatility of

absolute returns or tracking error, it follows that the investor's utility is conditioned on a single random variability — wealth in the case of absolute returns, and net wealth in the case of tracking error.

²DeGeorge, Patel, and Zeckhauser [1997] also find strong evidence that corporations actively manage their absolute and relative earnings performance, where relativity is defined by previous performance.

³Related to this, Tufano [1996] finds evidence that the amount of corporate risk management is related to the form of managerial compensation.

⁴DeMarzo and Duffie [1995] also argue that the extent of risk management activity can serve as a signal of corporate managerial skill.

⁵It is simple to show that our description of investor behavior is still consistent with the utility axioms of Von Neumann and Morgenstern [1947].

⁶It is important to note the difference between multiple dimensions of risk and multiple sources of uncertainty that underlie a single volatility measure.

⁷State Street Bank & Trust Company has recently started offering customized options similar to the disaster currency option discussed here.

⁸It should be noted that there is nothing unique about our zero demarcation lines. In fact, the critical boundaries between half-spaces can be specified as desired, as illustrated in Equation (4).

⁹At the expense of additional complexity, it is possible to extend our analysis to incorporate intertemporal default risk in the spirit of Rich [1996].

¹⁰The Wiener random variables are defined on the complete probability space $(\Omega, \mathcal{F}, \text{Pr})$. The correlations of the unanticipated portion of the returns are defined as $\rho_{i,j}(t) \equiv \text{corr}[dZ_i(t), dZ_j(t)]dt$, $\rho_{i,k}(t) \equiv \text{corr}[dZ_i(t), dZ_k(t)]dt$, and $\rho_{j,k}(t) \equiv \text{corr}[dZ_j(t), dZ_k(t)]dt$, where corr denotes the local correlation coefficient. The correlation coefficients are assumed constant during the life of the option (i.e., $\rho_{i,j}(t) \equiv \rho_{i,j}$, $\rho_{i,k}(t) \equiv \rho_{i,k}$, and $\rho_{j,k}(t) \equiv \rho_{j,k}$). As stated, it is also assumed that $\{Z_j(t), Z_i(t), Z_k(t)\}$ is a three-dimensional Brownian motion.

¹¹To derive this formula from Equation (4), let $S_j \equiv S_B$, $S_i \equiv S_p$, $\phi \equiv -1$, $S_k \equiv S_p$, $\sigma_j \equiv \sigma_B$, $\sigma_i \equiv \sigma_p$, $\rho_{i,j} \equiv \rho_{p,B}$, $\rho_{i,k} \equiv 1$, and $\rho_{j,k} \equiv \rho_{p,B}$. Further note that

$$1/2\sigma_p^2 - \sigma_p\sigma\bar{\rho} = -1/2\sigma_p^2 + \sigma_p\sigma_B\rho_{p,B}$$

¹²That is, $S_p = S_B = H = 100$, $r = 1.06$, $\sigma_p = 0.15$, $\sigma = 0.03$, $\bar{\rho} = 0.10$, and $d_p = d_B = T = 1$. This problem may also be stated in terms of the volatility of benchmark and the correlation of the portfolio and the benchmark rather than the tracking error and the correlation of the absolute and relative returns. It is easy to show

from $\sigma^2 = \sigma_p^2 + \sigma_B^2 - 2\sigma_p\sigma_B\rho_{p,B}$ and $\bar{\rho} = (\sigma_p - \sigma_B\rho_{p,B})/\sigma$ that if $\sigma_p = \sigma_B = 0.15$ and $\rho_{p,B} = 0.98$, then $\sigma = 0.03$ and $\bar{\rho} = 0.10$.

¹³Observe that as $H \rightarrow \infty$, y_2 and $y_3 \rightarrow -\infty$, and (4) converges to the Margrabe [1978] model.

¹⁴We recognize that the contingency may introduce a moral hazard to the writer because a manager who purchases the option would have an incentive to trip the option by penetrating the hurdle if absolute performance were close to the hurdle value and the relative performance was in the money. This problem, of course, will not apply to managers who replicate the contingent exchange option through dynamic trading.

¹⁵This formula can be derived from (4) by letting $S_j \equiv C$, $S_i \equiv X$, $\phi \equiv -1$, $S_k \equiv S$, $\sigma_j \equiv 0$, $\sigma_i \equiv \sigma_X$, $\rho_{i,j} \equiv 0$, $\rho_{i,k} \equiv \rho_{S,X}$, $\rho_{j,k} \equiv 0$, $d_i = r_p$, $d_j = r_d$, and $d_k = d_s$; noting that the tracking error collapses to the foreign currency volatility, $\sigma = \sigma_X$; and the correlation of absolute and relative returns is equal to the negative of the simple correlation, $\bar{\rho} = -\sigma_X\rho_{S,X}/\sigma_X = -\rho_{S,X}$.

¹⁶Bailey [1990] suggests the base fee should be viewed as a maintenance fee that “keeps the manager ‘in the game’ financially during periods of poor performance.” Grinold and Rudd [1987] discuss alternative forms of performance-based incentive fees.

¹⁷Record and Tynan [1987] first proposed combining absolute and relative measures of performance in the absence of a direct relative benchmark. We are aware of at least one institutional investment advisor whose incentive compensation is tied to more than one measure of return.

¹⁸This formula can be derived from (4) by specifying $\phi \equiv 1$, $S_j \equiv S_k \equiv S_p$, $S_i \equiv S_B$, $\sigma_j \equiv \sigma_k = \sigma_p$, $\sigma_i = \sigma_B$, $d_j = d_k = d_p$, $d_i = d_B$, $\rho_{i,j} = \rho_{p,B}$, $\rho_{j,k} = \rho_{p,B}$, and $\rho_{i,k} = \rho_{p,B}$.

¹⁹That is, $S_p = S_B = H = 100$, $\sigma_p = 0.15$, $\bar{\rho} = 0.10$, $\sigma = 0.03$, $r = 1.06$, and $d_B = d_p = T = 1$.

²⁰Specifically, the rate of change of the standard incentive fee with respect to a change in σ_p equals

$$S_p d_p^{-T} n(y_1) \sqrt{T} \frac{d\sigma}{d\sigma_p}$$

²¹The threat of termination and damage to one's reputation also serve as a control mechanism.

²²A zero-cost collar is possible with traditional portfolio insurance because after the put is purchased one degree of freedom remains, the strike price of the short call option. The exercise price, however, is stochastic in the contingent exchange option. Therefore, we use the hurdle value as our degree of freedom.

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