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OPTIMAL HEDGE FUND ALLOCATIONS: DO HIGHER MOMENTS MATTER?

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Abstract

Hedge funds have return peculiarities not commonly associated with traditional investment vehicles. Specifically, hedge funds seem more inclined to produce return distributions with significantly non-normal skewness and kurtosis.

There is also growing acceptance of the notion that investor preferences are better represented by bilinear utility functions or S-shaped value functions than by neo-classical utility functions such as power utility. Many investors have therefore concluded that mean-variance optimization is not appropriate for forming portfolios that include hedge funds.

We apply both mean-variance and full-scale optimization to form portfolios of hedge funds, given a wide range of assumptions about investor preferences. We find that higher moments of hedge funds do not meaningfully compromise the efficacy of mean-variance optimization if investors have power utility. We also find, however, that mean-variance optimization is not particularly effective for identifying optimal hedge fund allocations if preferences are bilinear or S-shaped. Finally, we show that investors with bilinear utility dislike kurtosis and that, contrary to conventional wisdom, investors with S-shaped preferences are attracted to kurtosis as well as negative skewness. Mean-variance optimization is insensitive to these preferences.

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I. Introduction

Hedge funds have return peculiarities not commonly associated with traditional investment vehicles. Specifically, hedge funds seem more inclined to produce return distributions with significantly non-normal skewness and kurtosis.

There is also growing acceptance of the notion that investor preferences are better represented by bilinear utility functions or S-shaped value functions than by neo-classical utility functions such as power utility. Many investors have therefore concluded that mean-variance optimization is not appropriate for forming portfolios that include hedge funds.

We apply both mean-variance and full-scale optimization to form portfolios of hedge funds, given a wide range of assumptions about investor preferences. We find that higher moments of hedge funds do not meaningfully compromise the efficacy of mean-variance optimization if investors have power utility. We also find, however, that mean-variance optimization is not particularly effective for identifying optimal hedge fund allocations if preferences are bilinear or S-shaped. In particular, we show that investors with bilinear utility dislike kurtosis and that, contrary to conventional wisdom, investors with S-shaped preferences are attracted to kurtosis as well as negative skewness. Mean-variance optimization is insensitive to these preferences.

We organize the paper as follows: In Part I we describe our data and summarize the distributional features of several hedge fund strategies. In Part II, we describe our methodology. Specifically, we compare mean-variance optimization to full-scale optimization, and we describe log wealth utility, bilinear utility, and S-shaped value functions. We present our results in Part III and summarize the paper in Part IV.

Part I: Statistical Summary of Hedge Fund Returns

We use monthly hedge fund returns for the 10-year period from January 1994 through December 2003, provided by the Center for International Securities and Derivatives Markets (CISDM), which is associated with the Isenberg School of Management at the University of Massachusetts. The CISDM database includes 2,500 funds with performance for some funds dating back to the early 1970's. We include only live funds with 10 years of history for four hedge fund strategies: equity hedge, convertible arbitrage, event driven, and merger arbitrage.

There are 25 funds in the equity hedge category that meet our criteria. These funds maintain a long position in equities perceived to be undervalued and hedge these holdings by selling short individual stocks or stock index futures perceived to be overvalued or neutrally valued. Often the managers of these funds employ leverage in order to raise the expected return. These funds tend to experience more extreme outcomes, which results in a higher level of kurtosis than a normal distribution.

We include 10 convertible arbitrage funds in our study. These funds typically purchase convertible bonds and hedge the equity risk by selling short the company's underlying common stock. These funds display both higher levels of kurtosis than a normal distribution and negative skewness. The negative skewness implies that these funds produce a greater number of above-average returns than a normal distribution, but negative outcomes tend to be lower than what we would expect from a normal distribution.

We also include 19 event driven funds. This hedge fund strategy is also referred to as corporate life cycle investing. These funds search for opportunities created by events such as spin-offs, mergers and acquisitions, bankruptcy reorganizations, recapitalizations, and share repurchases. They employ long and short positions in common and preferred stock, debt securities, and options. Because they attempt to capitalize on significant events, their return distributions display higher than normal kurtosis. They are also negatively skewed on average.

Finally, we include 7 merger arbitrage funds. These funds attempt to profit by acquiring the stock of companies that are takeover targets and by selling the stock of the acquiring companies. These funds also display higher than normal kurtosis.

Table 1 shows the skewness and kurtosis for each of these funds, and it indicates whether or not the funds passed the Jarque Bera test for normality. [1] A normal distribution has skewness equal to 0 and kurtosis equal to 3.

[1] The Jarque-Bera test is a non-parametric test of normality that is based on skewness and kurtosis. It is non-parametric in the sense that it tests for normality without specification of a particular mean or variance. The actual statistic is a sum of two independent parts, one built from the third moment, the other from the fourth moment. Each of these parts is distributed as a standard normal. The Jarque- Bera statistic is their sum of squares and thus should be compared to a chi-square distribution with two degrees of freedom.

Table 1: Higher Moments of Hedge Funds

Equity Hedge	Skewness	Kurtosis	JB Test
1	-0.25	4.98	Failed
2	-0.14	3.61	Passed
3	-0.34	4.69	Failed
4	0.43	4.77	Failed
5	0.07	5.32	Failed
6	0.00	4.75	Failed
7	1.12	8.23	Failed
8	-0.77	4.73	Failed
9	-0.70	5.13	Failed
10	-0.42	3.70	Passed
11	0.38	2.93	Passed
12	0.27	4.23	Failed
13	3.40	26.91	Failed
14	-0.87	6.33	Failed
15	-0.15	3.36	Passed
16	-0.17	6.35	Failed
17	-0.02	5.46	Failed
18	0.49	6.35	Failed
19	-0.42	6.46	Failed
20	-0.22	5.72	Failed
21	1.08	7.36	Failed
22	0.56	4.08	Failed
23	0.36	3.46	Passed
24	0.91	8.70	Failed
25	0.10	5.01	Failed

Convertible Arbitrage	Skewness	Kurtosis	JB Test
1	-0.45	3.79	Failed
2	-0.39	6.07	Failed
3	-0.84	6.22	Failed
4	-0.11	4.08	Passed
5	-0.35	4.78	Failed
6	0.19	4.44	Failed
7	-0.65	5.04	Failed
8	-0.60	5.25	Failed
9	-1.52	5.83	Failed
10	-1.34	8.72	Failed

Event Driven	Skewness	Kurtosis	JB Test
1	-1.00	7.60	Failed
2	0.00	4.18	Failed
3	1.43	14.72	Failed
4	0.91	6.87	Failed
5	-1.72	8.89	Failed
6	1.02	8.53	Failed
7	0.16	5.07	Failed
8	-0.50	4.88	Failed
9	0.29	3.92	Passed
10	-0.51	4.57	Failed
11	-0.66	4.28	Failed
12	-0.09	8.26	Failed
13	-0.37	3.75	Passed
14	-3.45	26.50	Failed
15	-0.46	6.26	Failed
16	-1.45	8.12	Failed
17	-0.09	3.98	Passed
18	-0.42	4.53	Failed
19	-0.14	5.30	Failed

Merger Arbitrage	Skewness	Kurtosis	JB Test
1	-0.30	4.10	Failed
2	0.91	8.40	Failed
3	-0.49	5.25	Failed
4	0.95	6.20	Failed
5	-1.01	6.97	Failed
6	0.17	4.92	Failed
7	0.88	9.75	Failed

Summary	Average Skewness	Average Kurtosis	Percent Failing JB
Equity Hedge	0.19	6.10	80%
Convertible Arbitrage	-0.61	5.42	90%
Event Driven	-0.37	7.38	84%
Merger Arbitrage	0.16	6.51	100%
All Hedge Funds	-0.12	6.44	85%

Part II. Methodology

We evaluate two approaches to portfolio formation: mean-variance optimization and full-scale optimization. Institutional investors typically employ mean-variance optimization to form portfolios, in part, because it requires knowledge only of the expected returns, standard deviations, and correlations of the portfolio's components. In principal, this approach is adequate if at least one of two conditions prevails. Either investors have quadratic utility or portfolio returns are normally distributed. Neither of these assumptions is literally true. Quadratic utility assumes that investors are equally averse to deviations above the mean as they are to deviations below the mean and that they sometimes prefer less wealth to more wealth. And as we have just shown, hedge fund returns exhibit significant departures from normality.

Computational efficiency now allows us to perform full-scale optimization as an alternative to mean-variance optimization. With this approach we calculate a portfolio's utility for every period in our sample considering as many asset mixes as necessary in order to identify the weights that yield the highest expected utility, given any utility function or S-shaped value function.

Suppose, for example, we wish to find the optimal blend between two funds whose returns are displayed in Table 2, assuming the investor has log wealth utility. We compute utility each period as $\ln[(1+R_A) \times W_A + (1+R_B) \times W_B]$, where R_A and R_B equal the returns of funds A and B, and W_A and W_B equal their respective weights.

We then shift the fund's weights using a numerical search procedure until we find the combination that maximizes expected utility, which for this example equals a 57.13% allocation to fund A and a 42.87% allocation to fund B. The expected utility of the portfolio reaches a maximum at 9.31%. This approach implicitly takes into account all of the features of the empirical sample, including skewness, kurtosis, and any other peculiarities of the distribution.

Table 2: Full-Scale Optimization for Log Wealth Investor							
Year	Fund A Returns	Fund B Returns	Fund A Weight	Fund B Weight	Portfolio Utility		
1	10.06%	16.16%	57.13%	42.87%	ln[(1+.1006) x .5713 + (1+.1616) x .4287] x 1/10	=	1.1931%
2	1.32%	-7.10%	57.13%	42.87%	ln[(1+.0132) x .5713 + (1 -.0710) x .4287] x 1/10	=	-0.2317%
3	37.53%	29.95%	57.13%	42.87%	ln[(1+.3753) x .5713 + (1+.2995) x .4287] x 1/10	=	2.9477%
4	22.93%	0.14%	57.13%	42.87%	ln[(1+.2293) x .5713 + (1+.0014) x .4287] x 1/10	=	1.2367%
5	33.34%	14.52%	57.13%	42.87%	ln[(1+.3334) x .5713 + (1+.1452) x .4287] x 1/10	=	2.2533%
6	28.60%	11.76%	57.13%	42.87%	ln[(1+.2860) x .5713 + (1+.1176) x .4287] x 1/10	=	1.9375%
7	20.89%	-7.64%	57.13%	42.87%	ln[(1+.2089) x .5713 + (1 -.0764) x .4287] x 1/10	=	0.8305%
8	-9.09%	16.14%	57.13%	42.87%	ln[(1 -.0909) x .5713 + (1+.1614) x .4287] x 1/10	=	0.1715%
9	-11.94%	7.26%	57.13%	42.87%	ln[(1 -.1194) x .5713 + (1+.0726) x .4287] x 1/10	=	-0.3777%
10	-22.10%	14.83%	57.13%	42.87%	ln[(1 -.2210) x .5713 + (1+.1483) x .4287] x 1/10	=	-0.6470%
Expected Utility							9.3138%

We apply these two portfolio formation techniques, mean-variance and full-scale optimization, to identify optimal portfolios, assuming five descriptions of investor preferences:

1. log wealth utility
2. bilinear utility with a -1% threshold
3. bilinear utility with a -5% threshold
4. S-shaped value function with aversion to returns below 0%
5. S-shaped value function with aversion to returns below 0.5%

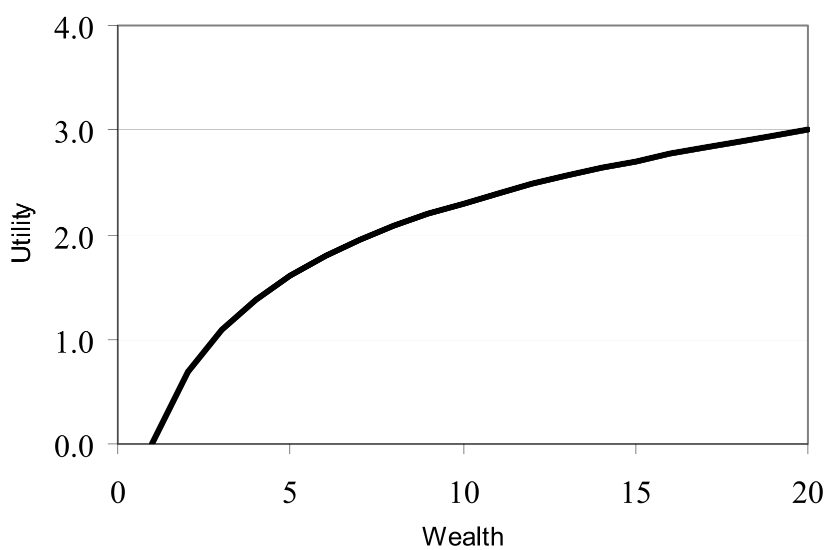
Log wealth utility was first proposed by Daniel Bernoulli in 1738.[2] This utility function assumes utility is equal to the natural logarithm of wealth, and resides in a broader family of utility functions called power utility. Power utility defines utility as $1/\gamma$ x Wealth^{γ}. A log wealth utility function is a special case of power utility. As γ

[2] Bernoulli, Daniel, "Exposition of a New Theory on the Measurement of Risk," *Econometrica*, January 1954 (Translation from 1738 version).

approaches 0, utility approaches the natural logarithm of wealth. A γ equal to ½ implies less risk aversion than log wealth, while a γ equal to -1 implies greater risk aversion.[3]

This class of utility functions assumes that investors prefer to expose the same proportion of their wealth to risk regardless of how much wealth they have, which is to say they have constant relative risk aversion. Figure 1 shows a log wealth utility function.

Figure 1: Log Wealth Utility Function



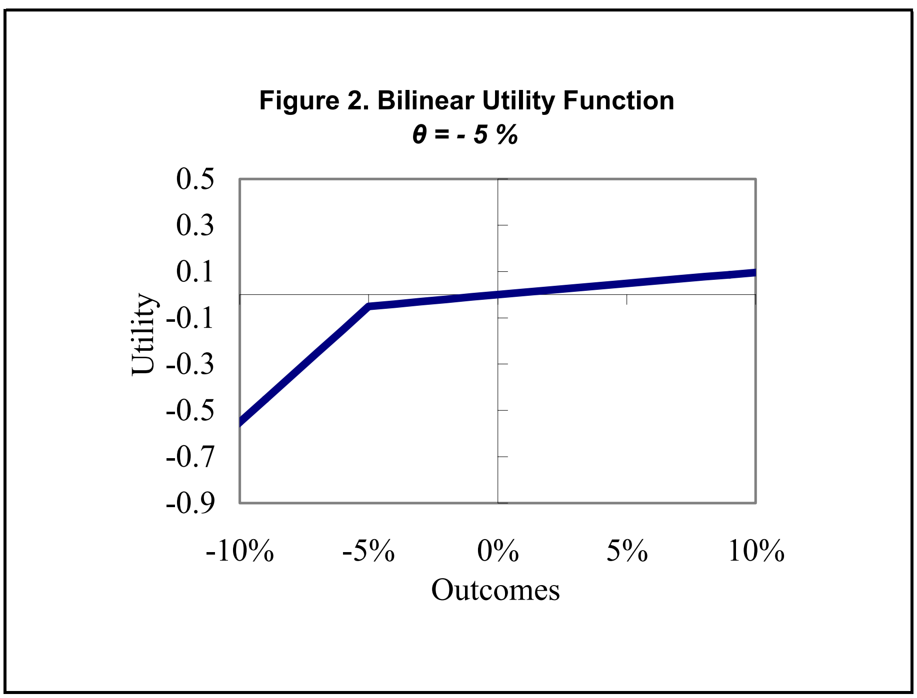
A bilinear utility function changes abruptly at a particular wealth or return level and is relevant for investors who are concerned with breaching a threshold. Consider, for example, a situation in which an investor requires a minimum level of wealth to maintain a certain standard of living. The investor's lifestyle might change drastically if she

[3] When γ equals -1, utility is expressed as $1 - W^{-1}$.

penetrates this threshold. Or she may be faced with a situation in which she will become insolvent if her wealth breaches some threshold, or a particular decline in wealth may breach a covenant on a loan. In these and similar situations, a bilinear utility function as described below is more likely to describe one's attitude toward risk.

$$U(x) = \begin{cases} \ln(1+x), & \text{for } x \geq \theta \\ 10 \times (x+\theta) + \ln(1-\theta), & \text{for } x < \theta \end{cases}$$

The kink is located at θ, which in one case we set equal to a monthly return of -1%, corresponding to about an 11% annual loss, and in the second case we set equal to a -5% monthly return, corresponding to a 46% annual loss. The first loss amount would be meaningful to most investors, whereas the second would be catastrophic. Figure 2 shows a bilinear utility function with the kink located at -5%:

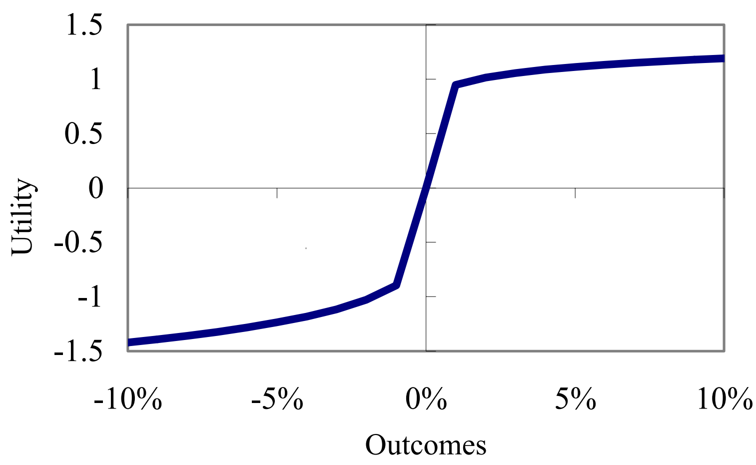


Proponents of behavioral finance have documented a number of contradictions to the neo-classical view of expected utility maximization. In particular, Kahnemann and Tversky (1979) have found that people focus on returns more than wealth levels and that they are risk averse in the domain of gains but risk seeking in the domain of losses. For example, if a typical investor is confronted with a choice between a certain gain and an uncertain outcome with a higher expected value, he will choose the certain gain. In contrast, when confronted with a choice between a certain loss and an uncertain outcome with a lower expected value, he will choose the uncertain outcome. This behavior is captured by an S-shaped value function, which Kahnemann and Tversky modeled as follows.

$$U(x) = \begin{cases} -A(\theta - x)^{\gamma_1}, & for\ x \leq \theta \\ +B(x-\theta)^{\gamma_2}, & for\ x > \theta \end{cases}$$

The portfolio's return is represented by *x*, and *A* and *B* are parameters that together control the degree of loss aversion and the curvature of the function for outcomes above and below the loss threshold, θ. In our analysis, we scale the parameters to match the range of returns one might reasonably expect from investment in hedge funds. In one case, we set the monthly loss threshold at 0%, which implies investors experience absolute loss aversion. In the second case, we set the monthly loss threshold equal to 0.5%, which corresponds to about a 6% annualized return. This higher threshold implies that investors experience loss aversion relative to a target return such as an actuarial interest rate assumption or a measure of purchasing power. Figure 3 shows an S-shaped value function with a loss threshold of 0%.

Figure 3. S-Shaped Value Function



Parameters

A	B	θ	γ_1	γ_2
2.25	1.50	1.00	0.20	0.10

Before we proceed with the optimizations, we scale each of the monthly hedge fund returns by a constant in order to produce means that conform to the implied returns of equally weighted portfolios. We assume that returns implied by reasonable weights better reflect investor expectations than historical hedge fund returns, in light of the well documented problems of survivorship bias and back fill bias.[4] This adjustment does not affect our comparisons, because we apply it to both the mean-variance and full-scale optimizations; hence the differences we find arise solely from the higher moments of the distributions and not their means.

[4] This bias arises from the greater likelihood of managers to submit performance records from funds that have performed favorably than from funds that have performed poorly.

We then proceed by identifying portfolios of hedge funds in each style category and across the entire sample of funds that maximize utility for each of these utility functions, based on full-scale optimization. This approach reveals the true utility-maximizing portfolios given the precise shape of the empirical return distributions.

Next we apply mean-variance optimization to generate the efficient frontier of hedge funds in each category and across the entire sample. Within each category we evaluate the mean-variance efficient portfolio that has the same expected return as the true utility-maximizing portfolio.

Part III. Results

We employ several metrics to evaluate the efficacy of mean-variance optimization. For all utility functions we show the percentage difference in utility between the optimal portfolio determined by full-scale optimization and the mean-variance efficient portfolio with the same expected return. We also show for all utility functions the fraction of the portfolio that one would need to turn over in order to move from the mean-variance efficient portfolio to the true utility-maximizing portfolio.

For the bilinear utility functions we show the fraction of returns that exceed the kinks, and for the S-shaped value functions, the fraction of returns that exceed the inflection points, for both the mean-variance efficient portfolios and the true utility-maximizing portfolios.

Our final set of results shows the higher moments of the mean-variance and true utility-maximizing portfolios. These results clearly demonstrate the superiority of full-scale optimization to mean-variance optimization when higher moments matter.

Improvement in Utility

Table 2 shows the percentage change in utility gained by shifting from the mean-variance efficient portfolio to the true utility-maximizing portfolio determined by full-scale optimization. For investors with log wealth utility, mean-variance optimization closely matches the results of full-scale optimization. Other studies show that this result prevails for other variations of power utility as well.[5] Mean-variance optimization performs well in these situations because power utility is relatively insensitive to higher moments.

This result, however, does not prevail for investors who have bilinear utility or S-Shaped value functions. For investors with these preferences, mean-variance optimization results in significant loss of utility.

Table 2: Percentage Difference in Utility

	Log Wealth	Bilinear Utility		S-Shaped Value Functions	
	Utility	At -5%	At -1%	At 0%	At +0.5%
Equity Hedge	0.01%	20.93%	11.61%	30.76%	61.28%
Convertible Arbitrage	0.00%	4.37%	2.90%	9.67%	14.55%
Event Driven	0.00%	17.13%	32.62%	12.58%	23.65%
Merger Arbitrage	0.00%	7.51%	26.39%	2.41%	6.98%
All Hedge Funds	0.01%	25.77%	30.41%	13.53%	59.13%

Required Turnover

Table 3 depicts the fraction of the mean-variance efficient portfolio that would need to be traded in order to invest the portfolio in accordance with the full-scale optimal

[5] See, for example, Levy, Haim and Harry M. Markowitz, "Approximating Expected Utility by a Function of Mean and Variance," *American Economic Review*, June 1979, Vol. 69, No. 3, and Cremers, J, M. Kritzman, and S. Page, "Portfolio Formation with Higher Moments and Plausible Utility, "*Revere Street Working Paper Series, Financial Economics 272-12*, November 22, 2003.

weights. Again, we find that mean-variance optimization performs well for log wealth investors, except in the case in which hedge funds across all four styles are considered. Even in this case, though, the 15% departure from the optimal full-scale weights results in only slight utility loss. In contrast, the mean-variance exposures for investors with bilinear utility or S-shaped value functions differ substantially from the true utility maximizing weights.

Table 3: Turnover Required to Shift from Mean-Variance to Full-Scale Efficiency

	Log Wealth Utility	Bilinear Utility		S-Shaped Value Functions	
		At -5%	At -1%	At 0%	At +0.5%
Equity Hedge	4%	21%	31%	40%	53%
Convertible Arbitrage	0%	66%	63%	56%	40%
Event Driven	0%	84%	30%	31%	42%
Merger Arbitrage	0%	36%	42%	12%	14%
All Hedge Funds	15%	34%	56%	57%	55%

Success Rate

Next we show the rate at which the mean-variance and full-scale portfolios succeed in producing outcomes above the critical levels. Again, for the bilinear utility function, the critical levels equal -1% and -5%, the location of the kinks. For the S-shaped value functions, the critical levels correspond to the returns at which the inflection point is located, which equal 0% and 0.5%.

Table 4: Success Rates										
	Log Wealth Utility		Bilinear Utility				S-Shaped Value Functions			
			At -5 %		At -1 %		At 0 %		At + 0.5 %	
	MV	FS	MV	FS	MV	FS	MV	FS	MV	FS
Equity Hedge	NA	NA	99%	100%	96%	97%	75%	88%	63%	77%
Convertible Arbitrage	NA	NA	82%	75%	98%	98%	89%	95%	79%	86%
Event Driven	NA	NA	99%	99%	95%	98%	88%	94%	78%	87%
Merger Arbitrage	NA	NA	97%	98%	97%	98%	95%	97%	83%	86%
All Hedge Funds	NA	NA	99%	99%	97%	97%	84%	91%	65%	78%

There is no significant difference between the success rate of the full-scale portfolios and the mean-variance portfolios for the bilinear utility function. The most significant differences occur when investors are presumed to have an S-shaped value function with the inflection point set at 0.5%. In general, the full-scale portfolios perform much better than the mean-variance portfolios across all hedge fund strategies.

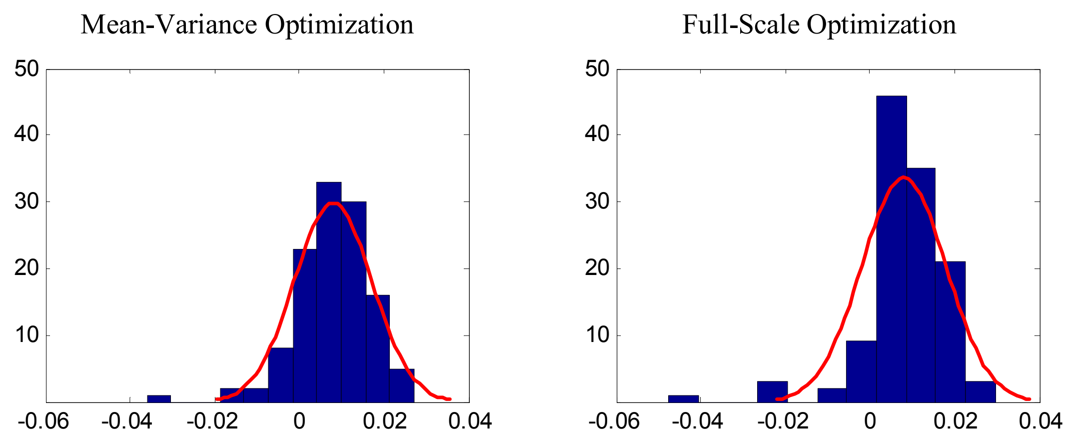
Skewness

Table 5 shows the skewness of the mean-variance and full-scale portfolios. On average, full-scale optimization reduces negative skewness when preferences are bilinear and increases it when they are S-shaped. The bilinear utility functions have a steep slope below the kink; hence significantly negative outcomes are sharply penalized. In contrast, the S-shaped value functions do not penalize severely negative outcomes much more than marginally negative outcomes, and they place a premium on above-threshold outcomes.

Table 5: Skewness										
	Log Wealth Utility		Bilinear Utility				S-Shaped Value Functions			
			At -5 %		At -1 %		At 0 %		At + 0.5 %	
	MV	FS	MV	FS	MV	FS	MV	FS	MV	FS
Equity Hedge	-0.23	-0.08	-0.30	-0.01	0.16	0.39	-0.08	-0.34	-0.20	-0.34
Convertible Arbitrage	4.78	4.78	-0.39	0.03	-0.39	0.20	-0.40	-0.32	-0.38	-0.80
Event Driven	1.02	1.02	-1.08	-0.24	-1.48	-0.60	-1.58	-1.82	-1.46	-1.85
Merger Arbitrage	0.17	0.17	-0.05	0.04	-1.09	-0.06	-1.02	-1.10	-1.22	-1.60
All Hedge Funds	-0.56	-0.55	0.01	0.01	-1.12	0.30	-1.10	-1.98	-0.95	-1.12

Figure 4 contrasts the mean-variance and full-scale return distributions for investors with S-shaped preferences, based on the entire sample of hedge funds. It clearly shows that full-scale optimization increases negative skewness compared to mean-variance optimization.

Figure 4: S-Shaped Distribution, $\theta = 0$



Kurtosis

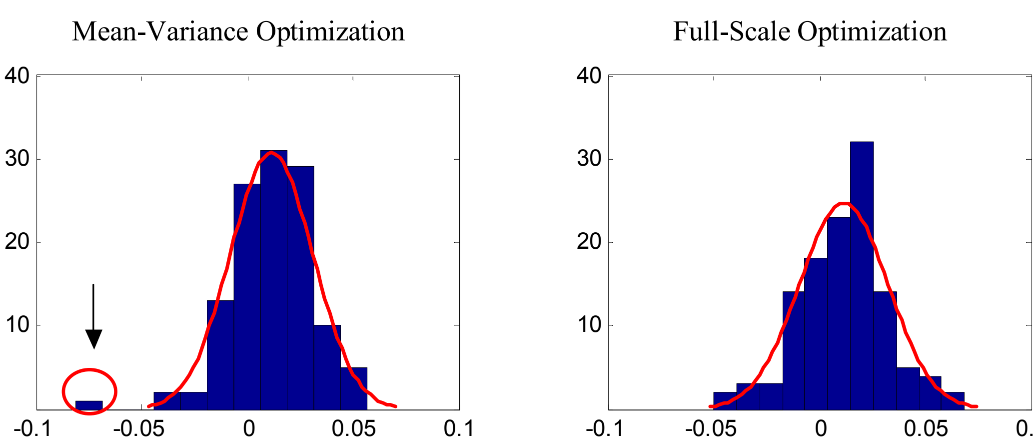
Finally, we report the kurtosis of the mean-variance and full-scale portfolios, and again we see a contrast in the impact of full-scale optimization. Full scale-optimization reduces kurtosis given bilinear utility but not if investors have S-shaped value functions. Bilinear utility penalizes incrementally more negative results more severely than S-shaped value functions, and high kurtosis generates a high frequency of significantly negative results.

Table 6: Kurtosis

	Log Wealth Utility		Bilinear Utility				S-Shaped Value Functions			
			At -5 %		At -1 %		At 0 %		At + 0.5 %	
	MV	FS	MV	FS	MV	FS	MV	FS	MV	FS
Equity Hedge	4.70	4.65	4.17	3.65	2.39	2.44	2.86	3.80	3.28	4.44
Convertible Arbitrage	4.78	4.78	5.34	4.67	4.18	3.41	5.38	5.88	4.64	5.62
Event Driven	8.53	8.53	7.51	4.06	9.30	4.98	9.24	12.56	9.18	11.67
Merger Arbitrage	4.92	4.92	4.36	3.95	6.13	4.12	7.59	8.70	7.07	8.95
All Hedge Funds	3.86	3.85	6.46	3.62	6.55	2.98	6.62	11.08	6.22	8.47

Figure 5 compares the return distributions produced by mean-variance and full-scale optimization for investors with bilinear utility, again based on the entire sample of hedge funds. It shows that full-scale optimization reduces the kurtosis associated with negative returns, but not the kurtosis associated with positive returns.

Figure 5: Bilinear Distribution, θ = -5%



IV. Summary

Investors recognize that many hedge fund strategies produce significantly non-normal return distributions, which calls into question the efficacy of using mean-variance optimization to form portfolios that include hedge funds. We directly measure the efficacy of mean-variance optimization by comparing it to full-scale optimization, given a variety of assumptions about investor preferences. Our analysis yields several insights.

- Mean-variance optimization performs extremely well for investors with log wealth utility. This result prevails even though the distributions of the component hedge fund returns are significantly non-normal. Moreover, much of this non-normality survives into the mean-variance efficient portfolios, which implies that log wealth utility is fairly insensitive to higher moments.
- Mean-variance optimization performs poorly for investors with bilinear utility or S-shaped value functions. Full-scale optimization reduces kurtosis and negative skewness for investors with bilinear utility, and it maximizes the success rate for investors with S-shaped value functions.
- Negative skewness is not problematic, given S-shaped value functions, because it raises the success rate and because extreme negative results are not severely penalized.
- Kurtosis is also not problematic for investors with S-shaped value functions, again because these functions do not severely penalize extreme negative results.

These insights, of course, depend on the particular parameters chosen for the bilinear utility functions and S-shaped value functions. Different parameters may lead to different conclusions about the sensitivity of utility to skewness and kurtosis. In any event, it is safe to assume that mean-variance optimization is suitable for investors with log wealth utility and other variations of power utility, even when allocating among funds with significantly non-normal distributions. However, we strongly recommend that investors with bilinear utility or S-shaped value functions employ full-scale optimization when forming portfolios that include hedge funds or other assets with non-normal distributions.

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Appendix

Tables A1 through A5 show the specific weights resulting from mean-variance and full-scale optimization. By comparing these weights to the higher moment measures shown in Table 1, it is possible to gain insight about sensitivity of the various utility and value functions to higher moments.

Table A1: Equity Hedge Optimal Portfolio Weights

	Log Wealth		Bilinear at - 5 %		Bilinear at - 1 %		S-Shaped at 0 %		S-Shaped at +0.5 %	
Fund	MV	FS	MV	FS	MV	FS	MV	FS	MV	FS
1	1.3%	1.7%	3.6%	3.8%	0.4%	0.0%	2%	0.0%	3.0%	23.0%
2	0.0%	0.0%	4.5%	5.3%	6.9%	9.4%	7%	6.4%	6.3%	3.8%
3	0.0%	0.0%	5.6%	4.1%	14.5%	16.6%	10%	4.2%	8.4%	19.7%
4	5.5%	5.5%	0.0%	4.5%	0.0%	0.0%	0%	0.0%	0.0%	0.1%
5	9.7%	9.6%	2.7%	2.9%	0.0%	0.0%	0%	0.0%	0.2%	0.0%
6	0.0%	0.0%	11.4%	4.7%	13.4%	6.7%	13%	21.0%	12.4%	23.4%
7	0.0%	0.0%	6.2%	4.3%	12.7%	10.7%	13%	7.7%	11.8%	0.0%
8	7.1%	6.6%	3.1%	2.8%	0.0%	0.0%	1%	4.8%	1.5%	3.2%
9	0.0%	0.0%	11.3%	4.5%	30.7%	14.1%	24%	9.6%	20.7%	4.6%
10	4.7%	4.1%	3.8%	2.9%	0.9%	1.1%	2%	2.8%	2.9%	4.2%
11	0.0%	0.0%	3.7%	4.7%	1.4%	4.7%	3%	0.0%	3.0%	0.0%
12	4.9%	5.1%	3.6%	3.3%	0.0%	0.0%	0%	6.2%	1.6%	1.5%
13	0.9%	2.4%	3.5%	5.4%	0.0%	1.2%	2%	1.9%	2.7%	0.0%
14	0.0%	0.0%	0.0%	4.5%	0.0%	13.1%	0%	11.0%	0.0%	1.4%
15	15.3%	15.6%	2.2%	3.5%	0.0%	0.0%	0%	0.0%	0.0%	0.0%
16	0.0%	0.0%	3.2%	3.2%	0.0%	1.7%	0%	0.0%	0.5%	0.0%
17	1.8%	2.4%	4.3%	3.2%	3.4%	0.5%	5%	8.7%	5.3%	0.0%
18	0.0%	0.0%	4.2%	7.1%	10.1%	10.6%	7%	7.9%	6.0%	4.8%
19	3.4%	1.4%	3.1%	2.6%	0.0%	0.1%	1%	7.6%	1.9%	5.0%
20	8.4%	8.0%	1.9%	2.8%	0.0%	4.6%	0%	0.0%	0.0%	0.0%
21	0.0%	0.0%	4.6%	3.6%	3.6%	2.8%	5%	0.4%	5.8%	0.5%
22	11.2%	11.9%	2.8%	5.6%	0.0%	0.0%	0%	0.0%	0.1%	0.6%
23	4.8%	5.3%	4.1%	4.3%	0.4%	2.1%	2%	0.0%	3.0%	0.0%
24	11.7%	11.3%	3.1%	3.1%	0.0%	0.0%	0%	0.0%	0.0%	2.5%
25	9.1%	8.9%	3.4%	3.1%	1.6%	0.0%	2%	0.0%	2.7%	1.8%

Table A2: Convertible Arbitrage Optimal Portfolio Weights

	Log Wealth		Bilinear at - 5 %		Bilinear at - 1 %		S-Shaped at 0 %		S-Shaped at +0.5 %	
Fund	MV	FS	MV	FS	MV	FS	MV	FS	MV	FS
1	0.0%	0.0%	12.9%	1.0%	4.4%	2.8%	8.3%	19.4%	6.0%	7.8%
2	0.0%	0.0%	3.5%	0.0%	9.5%	0.0%	9.8%	21.7%	9.6%	11.9%
3	0.0%	0.0%	0.0%	0.2%	17.5%	35.5%	13.0%	3.5%	15.7%	3.5%
4	0.0%	0.0%	10.7%	0.0%	3.1%	0.0%	7.4%	0.0%	4.9%	11.1%
5	100.0%	100.0%	22.5%	19.2%	3.3%	0.0%	7.1%	2.7%	4.8%	1.5%
6	0.0%	0.0%	1.6%	61.4%	10.2%	51.4%	10.1%	37.1%	10.2%	13.1%
7	0.0%	0.0%	2.2%	0.0%	7.2%	2.2%	9.6%	1.5%	8.2%	25.4%
8	0.0%	0.0%	0.0%	6.2%	41.7%	4.7%	25.3%	0.0%	35.4%	11.8%
9	0.0%	0.0%	38.1%	12.1%	0.0%	3.4%	1.2%	0.0%	0.0%	5.1%
10	0.0%	0.0%	8.6%	0.0%	3.0%	0.0%	8.3%	14.1%	5.2%	8.8%

Table A3: Event Driven Optimal Portfolio Weights

	Log Wealth		Bilinear at - 5 %		Bilinear at - 1 %		S-Shaped at 0 %		S-Shaped at +0.5 %	
Fund	MV	FS	MV	FS	MV	FS	MV	FS	MV	FS
1	0.0%	0.0%	0.0%	0.0%	24.2%	16.0%	29.4%	39.9%	23.4%	25.4%
2	0.0%	0.0%	29.0%	0.0%	10.7%	9.4%	13.4%	5.7%	10.5%	11.6%
3	0.0%	0.0%	0.0%	18.4%	3.7%	1.1%	3.5%	0.0%	3.8%	0.8%
4	0.0%	0.0%	0.5%	10.7%	1.9%	12.8%	0.5%	0.4%	2.1%	0.5%
5	0.0%	0.0%	0.7%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	4.7%
6	100.0%	100.0%	0.0%	1.1%	3.0%	3.7%	1.8%	3.9%	3.1%	2.1%
7	0.0%	0.0%	2.1%	25.4%	0.3%	2.0%	0.0%	0.0%	0.6%	4.3%
8	0.0%	0.0%	1.0%	0.0%	6.2%	0.0%	4.2%	6.1%	6.2%	0.0%
9	0.0%	0.0%	9.4%	7.3%	4.2%	4.1%	3.1%	1.0%	4.3%	0.0%
10	0.0%	0.0%	0.0%	0.0%	2.1%	0.5%	0.4%	0.0%	2.3%	12.0%
11	0.0%	0.0%	0.0%	0.0%	7.7%	9.7%	7.5%	0.0%	7.6%	0.0%
12	0.0%	0.0%	15.7%	6.5%	11.1%	22.6%	12.5%	10.1%	10.8%	0.0%
13	0.0%	0.0%	0.0%	5.5%	0.4%	0.0%	0.0%	0.0%	0.6%	0.0%
14	0.0%	0.0%	4.6%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	10.8%
15	0.0%	0.0%	30.3%	0.0%	13.7%	11.2%	16.4%	32.7%	13.6%	18.6%
16	0.0%	0.0%	6.8%	0.0%	4.4%	0.0%	2.8%	0.0%	4.6%	6.0%
17	0.0%	0.0%	0.0%	7.0%	0.0%	2.2%	0.0%	0.0%	0.0%	3.3%
18	0.0%	0.0%	0.0%	16.0%	0.1%	0.9%	0.0%	0.0%	0.4%	0.0%
19	0.0%	0.0%	0.0%	2.2%	6.3%	3.7%	4.4%	0.0%	6.3%	0.0%

Table A4: Merger Arbitrage Optimal Portfolio Weights

	Log Wealth		Bilinear at - 5 %		Bilinear at - 1 %		S-Shaped at 0 %		S-Shaped at +0.5 %	
Fund	MV	FS	MV	FS	MV	FS	MV	FS	MV	FS
1	14.5%	13.6%	14.7%	11.7%	14.7%	0.0%	0.0%	0.0%	14.8%	21.6%
2	11.1%	0.0%	3.6%	0.0%	6.7%	18.5%	0.0%	0.0%	5.1%	0.0%
3	11.9%	14.3%	6.5%	1.7%	8.7%	0.2%	0.0%	0.0%	7.6%	7.4%
4	28.0%	0.0%	59.5%	62.2%	46.8%	75.1%	0.0%	0.0%	53.5%	45.6%
5	14.6%	52.7%	15.4%	24.0%	15.1%	0.0%	0.0%	0.0%	15.3%	17.2%
6	10.0%	7.9%	0.3%	0.0%	4.2%	5.8%	100.0%	100.0%	2.1%	0.9%
7	9.9%	11.5%	0.0%	0.3%	3.8%	0.3%	0.0%	0.0%	1.7%	7.2%

Table A5: All Funds Optimal Portfolio Weights

Fund	Log Wealth MV	Log Wealth FS	Bilinear at - 5 % MV	Bilinear at - 5 % FS	Bilinear at - 1 % MV	Bilinear at - 1 % FS	S-Shaped at 0 % MV	S-Shaped at 0 % FS	S-Shaped at +0.5 % MV	S-Shaped at +0.5 % FS
EH1	0.0%	0.0%	1.8%	0.1%	0.0%	0.0%	0.0%	0.0%	3.4%	0.0%
EH2	0.0%	0.0%	1.8%	6.7%	0.5%	8.4%	0.7%	3.1%	0.8%	2.1%
EH3	0.0%	0.0%	1.7%	2.1%	1.7%	8.0%	1.5%	1.8%	0.8%	1.4%
EH4	0.0%	0.0%	1.9%	1.8%	0.0%	0.0%	0.0%	0.2%	3.8%	0.0%
EH5	0.0%	0.0%	1.7%	0.3%	0.1%	0.0%	0.4%	0.0%	2.3%	1.4%
EH6	0.0%	0.0%	1.4%	2.0%	2.4%	1.8%	2.6%	2.3%	0.0%	0.0%
EH7	0.0%	0.0%	1.7%	2.3%	1.9%	3.0%	2.1%	0.1%	0.0%	0.0%
EH8	0.0%	0.0%	1.9%	0.0%	0.0%	0.0%	0.0%	2.3%	4.3%	8.6%
EH9	0.0%	0.0%	0.8%	1.1%	0.0%	2.9%	0.9%	1.6%	0.0%	1.2%
EH10	0.0%	0.0%	1.7%	0.6%	0.6%	0.0%	0.6%	0.0%	2.2%	0.0%
EH11	0.0%	0.0%	1.7%	4.0%	0.0%	6.8%	0.2%	0.0%	2.5%	0.0%
EH12	0.0%	0.0%	1.9%	1.9%	0.0%	0.0%	0.0%	0.4%	2.8%	0.0%
EH13	0.0%	0.0%	1.7%	4.4%	0.3%	4.1%	0.6%	0.0%	2.1%	0.0%
EH14	0.0%	0.0%	2.4%	1.1%	5.6%	2.5%	4.6%	1.6%	1.0%	0.2%
EH15	0.0%	0.0%	1.7%	3.3%	0.4%	0.0%	0.6%	0.0%	2.0%	0.0%
EH16	0.0%	0.0%	1.7%	0.6%	0.1%	0.0%	0.5%	0.2%	2.4%	0.2%
EH17	0.0%	0.0%	2.0%	2.5%	0.0%	0.5%	0.0%	0.0%	0.1%	0.0%
EH18	0.0%	0.0%	1.7%	4.6%	1.2%	2.5%	1.3%	2.2%	1.3%	2.6%
EH19	0.0%	0.0%	1.8%	0.0%	0.0%	0.0%	0.0%	1.7%	2.6%	4.9%
EH20	0.0%	0.0%	1.8%	0.0%	0.0%	0.0%	0.0%	1.5%	4.2%	3.6%
EH21	0.0%	0.0%	1.6%	1.9%	0.5%	0.0%	0.8%	2.0%	0.0%	2.0%
EH22	0.0%	0.0%	1.7%	5.0%	0.5%	2.4%	0.7%	0.0%	2.5%	0.0%
EH23	0.0%	0.0%	1.7%	2.0%	1.2%	1.8%	1.4%	0.0%	1.3%	0.0%
EH24	0.0%	0.0%	1.8%	0.0%	0.0%	0.0%	0.0%	0.0%	4.1%	5.5%
EH25	63.1%	64.6%	1.7%	5.2%	0.9%	0.0%	1.1%	0.0%	2.6%	2.4%
ALL EH	63.1%	64.6%	43.1%	53.6%	18.2%	44.7%	20.6%	21.1%	49.2%	36.1%
CA 1	0.0%	0.0%	1.5%	0.5%	0.7%	0.6%	1.2%	5.7%	0.0%	0.4%
CA2	0.0%	0.0%	1.8%	1.3%	0.0%	3.8%	0.0%	3.9%	0.0%	0.2%
CA3	0.0%	0.0%	1.3%	1.8%	6.7%	4.0%	5.6%	3.9%	0.0%	0.0%
CA4	0.0%	0.0%	1.3%	1.1%	3.0%	4.3%	3.0%	0.7%	0.0%	0.0%
CA5	0.0%	0.0%	1.7%	2.7%	0.3%	0.1%	0.6%	0.9%	3.4%	0.0%
CA6	0.0%	0.0%	0.8%	1.8%	8.1%	5.7%	7.6%	1.7%	0.0%	0.0%
CA7	0.0%	0.0%	1.3%	1.1%	0.0%	1.4%	0.0%	3.5%	0.0%	1.4%
CA8	0.0%	0.0%	0.0%	1.7%	15.2%	4.7%	14.2%	3.0%	0.0%	0.0%
CA9	0.0%	0.0%	2.6%	1.2%	0.0%	2.4%	0.0%	2.1%	3.1%	0.0%
CA10	0.0%	0.0%	2.2%	0.6%	0.0%	1.8%	0.0%	2.9%	0.0%	1.1%
ALL CD	0.0%	0.0%	3.7%	1.1%	0.7%	2.3%	1.2%	8.6%	0.0%	1.5%
ED 1	0.0%	0.0%	1.2%	0.7%	2.5%	1.7%	2.6%	4.2%	0.0%	1.6%
ED2	0.0%	0.0%	1.6%	2.3%	3.0%	3.6%	2.7%	4.7%	0.0%	0.0%
ED3	0.0%	0.0%	1.7%	1.1%	1.4%	1.8%	1.4%	0.0%	1.7%	2.0%
ED4	0.0%	0.0%	1.7%	1.0%	0.0%	2.6%	0.3%	0.2%	2.4%	0.0%
ED5	0.0%	0.0%	1.6%	0.3%	0.0%	0.0%	0.0%	0.8%	6.1%	4.7%
ED6	2.5%	2.5%	1.7%	0.0%	0.1%	0.0%	0.3%	0.4%	2.6%	10.5%
ED7	0.0%	0.0%	2.0%	2.9%	0.0%	0.0%	0.0%	0.0%	5.8%	0.0%
ED8	0.0%	0.0%	1.4%	0.3%	3.3%	0.0%	3.3%	3.5%	0.0%	0.1%
ED9	0.0%	0.0%	1.7%	1.7%	0.0%	0.3%	0.3%	0.0%	2.3%	2.8%
ED10	0.0%	0.0%	1.7%	1.2%	1.3%	0.4%	1.4%	2.3%	1.6%	2.4%
ED11	0.0%	0.0%	1.5%	1.0%	0.0%	0.6%	0.2%	2.3%	0.0%	3.5%
ED12	0.0%	0.0%	1.6%	1.8%	1.1%	3.3%	0.9%	3.2%	0.0%	0.0%
ED13	0.0%	0.0%	1.7%	1.2%	0.1%	0.0%	0.5%	0.0%	2.9%	0.0%
ED14	0.0%	0.0%	2.1%	0.0%	0.0%	0.0%	0.0%	3.9%	3.4%	4.2%
ED15	0.0%	0.0%	0.8%	0.7%	7.9%	1.8%	8.0%	3.3%	0.0%	2.2%
ED16	0.0%	0.0%	1.7%	0.8%	0.5%	0.0%	0.8%	3.1%	0.5%	0.0%
ED17	0.0%	0.0%	2.4%	1.8%	0.0%	0.0%	0.0%	1.1%	6.0%	1.7%
ED18	0.0%	0.0%	1.9%	1.9%	0.0%	0.0%	0.0%	0.0%	3.3%	0.0%
ED19	0.0%	0.0%	1.6%	1.5%	0.6%	2.5%	0.9%	1.3%	0.0%	0.0%
ALL ED	2.5%	2.5%	31.8%	22.2%	21.8%	18.6%	23.6%	34.2%	38.7%	35.7%
MA 1	0.0%	0.0%	2.0%	0.8%	1.1%	0.0%	1.1%	4.0%	0.0%	2.8%
MA2	0.0%	0.0%	1.7%	1.3%	0.0%	2.2%	0.0%	1.7%	0.4%	4.2%
MA3	0.0%	0.0%	1.8%	1.1%	0.0%	1.7%	0.0%	2.7%	0.0%	0.0%
MA4	0.0%	0.0%	0.0%	1.3%	23.2%	4.2%	20.3%	3.4%	0.0%	0.0%
MA5	0.0%	0.0%	1.6%	1.5%	0.5%	0.0%	0.8%	4.1%	0.0%	4.3%
MA6	34.4%	32.9%	1.8%	4.3%	0.3%	0.0%	0.5%	0.0%	3.2%	8.6%
MA7	0.0%	0.0%	1.7%	0.1%	0.8%	0.0%	1.0%	0.5%	2.1%	5.1%
ALL MA	34.4%	32.9%	10.6%	10.3%	26.0%	8.1%	23.7%	16.4%	5.6%	25.0%